ON RECESSION CURVES

1 — Recession Equations

C. Toebes and D. D. Strang*

SUMMARY

The various recession equations available in the literature are examined and some differences between them are noted. Methods are examined for developing master recession curves for surface flow, interflow and base flow; and the equations are applied to some actual cases. The authors conclude, with Indri (1960), that for practical purposes it is better to use an empirical recession equation if it fits the data than to use a simple theoretical one; and they recommend the use of five different equations.

INTRODUCTION

The recession curve or falling limb of a hydrograph represents the diminishing discharge from storage in the absence of further replenishment. Figure 1 is a simple representation of part of the hydrological cycle and indicates the various storages from which stream flow is supplied. All recessions represent withdrawal from storage modified by channel storage; the shape of each recession therefore depends not only on the nature and extent of the storage reservoir, but also on the nature and extent of the channels through which the flow is routed.

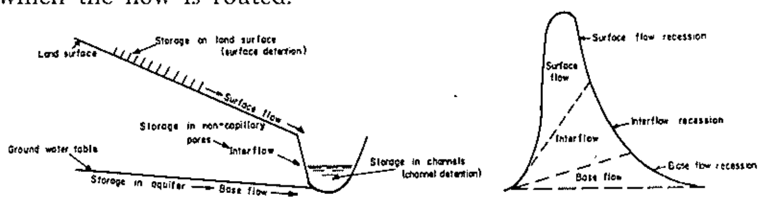

Fig. 1 — SIMPLE REPRESENTATION of part of the hydrological cycle and storm hydrograph.

Sometimes the interflow is indistinguishable and is combined with the surface flow, the combined flow being called direct flow with its associated direct flow recession. Sometimes bank storage is considered separately; it is the storage caused by flow into the banks of a stream during rising stages and will deplete during

* Soil Conservation and Rivers Control Branch, Ministry of Works, Wellington.

falling stages and may cause a recession similar to interflow. It is considered that bank storage from effluent streams (Fig. 2) is normally insignificant. However, it may become important when the point at which a stream changes from effluent to influent moves considerably down stream during or shortly after heavy rain as shown in Fig. 2.

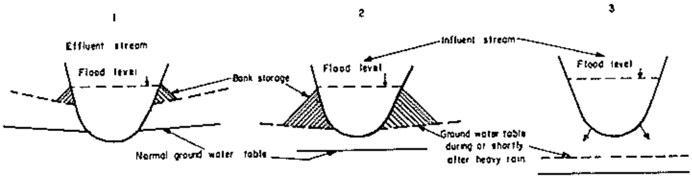

Fig. 2 — CHANGE OF STREAM CHANNEL from effluent to influent and relative importance of bank storage.

Much has been written on recessions; this is mainly due to the relative ease by which the recession can be represented mathematically. Maillet (1905) suggested the simple exponential equation

$$q_t = q_0 e^{-at}$$

where q_t is the discharge at time 't', q_o the initial discharge, 'e' the base of the natural logarithm and 'a' a constant. Normally e^{-a} is replaced by k, which is called the recession constant. This equation is the most widely used for expression of part or all of the entire recession.

The authors believe that the use of this equation is frequently an over-simplification resulting from a lack of data or from the method of plotting recessions, and they re-examine the equations available in the literature and apply these to some actual cases.

The results presented are based on catchments of up to 216 sq. miles with moderately steep, to steep, topography. They are not necessarily applicable to larger catchments but are likely to be suitable to most New Zealand ones.

EQUATIONS

Simple Exponential

Theoretical investigations and empirical studies have shown that either the entire recession curve, or part of it, can be expressed by

$$q_t = q_0 e^{-at} \quad \ldots\ldots (1)$$

where e^{-a} is normally replaced by k. The first theoretical derivation is ascribed to Maillet (1905). Werner and Sundquist (1951) show that the diminishing discharge from a confined aquifer (i.e. the base flow recession) can be expressed by Eq. (1). Horton (1933), Barnes (1939) and others have shown that part or all of the hydrograph recession can be fitted empirically by this equation.

If the recession conforms to the simple exponential equation then from

$$\log q_t = t \log k + \log q_0$$

it can be seen that a plot of time (natural scale) and discharge (log. scale) results in a straight line of slope log k.

Double Exponential

Horton (1933) suggested an improvement to the simple exponential by introducing a constant exponent 'n' to the time, thus

$$q_t = q_0 e^{-bt^n} \quad \text{.................................} \quad (2)$$

where $e^{-b} = k$ and n is a constant. So far as the authors have been able to ascertain Eq. (2) is purely empirical. Because

$$\log q_t = -b \, (\log e) \, t^n + \log q_c$$

then data following a double exponential equation will plot as a straight line for t^n (natural scale) and discharge (log. scale) provided a suitable 'n' is chosen. The value of 'b' is determined from the slope of the line. This method is described by Wisler and Brater (1949). An alternative method given by Johnson and Dils (1956) is to take logs a second time and obtain

$$\log \log \left(\frac{q_o}{q_t}\right) = n \log t + \log b - 0.36222$$

and to derive the constants by the method of least squares. The present authors plotted $\log \left(\frac{q_o}{q_t}\right)$ and 't' on log-log paper for a suitable q_0; this resulted in a straight line if the double exponential fitted the relevant recession. It is considered that by studying the hydrograph it is posible to estimate q_0 more quickly and more easily than 'n'.

Hyperbola

A third curve which is a hyperbola of the form

$$q_t = \frac{q_0}{(1 + ct)^2} \quad \text{.................................} \quad (3)$$

where 'c' is a constant, has been fitted to stream flow data by Maillet (1905). Werner and Sundquist (1951) derive this equation theoretically for out-flow from unconfirmed aquifers (i.e., base flow recession). Chapman (1963) puts Eq. (3) in the form:

$$\sqrt{\frac{1}{q_t}} - \sqrt{\frac{1}{q_0}} = k't$$

Written in logarithmic form (3) becomes

$$\log q_t = -2 \log (1 + ct) + \log q_0$$

This curve plots as a straight line on log-log paper for the variables q_t and $(1 + ct)$.

Ice Melt Hyperbola

An empirical equation which has been applied by Indri (1960) to low flows in the Venetian Alps is

$$q_t = \frac{1}{gt} + h$$

where 'g' and 'h' are constants.

A better equation could be achieved by including, in the style of Horton for the double exponential, an exponent 'n' so that:

$$q = \frac{a}{t^n} + b \quad \text{.........} \quad (4)$$

where 'a' and 'b' are constants. Because this curve asymptotically approaches a constant discharge it may typify base flow recessions in areas where permanent snow and ice are present and for this reason it is called an ice melt hyperbola. Since for this equation

$$\log (q_t - b) = -n \log t + \log a$$

a plot on log-log paper of $(q_t - b)$ and 't' will result in a straight line of slope $-n$, provided 'b' has been chosen correctly. A study of the stream flow data should give an approximate estimate of a constant 'b' for initial trials.

Ice Melt Exponential

Some writers in attempting to achieve a better fit to hydrograph recessions have advocated various alterations to the simple exponential equation. One form proposed by Wicht (1943) is of the form

$$q_t = a + (q_0 - a)\, k^t \quad \text{.........} \quad (5)$$

where 'a' and 'k' are constants. This equation is similar to (4) in that for large 't' the discharge asymptotically approaches a constant value. Since in many cases the discharge, which is largely supplied by melt from permanent ice and snowfields, reaches an apparent constant value for some finite duration, then this equation, like the ice melt hyperbola is suitable for catchments strongly influenced by snow and ice — provided one is cautious not to extend such curves too far in time. However, unlike (4) the curve is relatively flat for small 't'. The logarithmic form of (5) is

$$\log (q_t - a) = t \log k + \log (q_0 - a)$$

hence for a suitable constant 'a', a plot of $q_t - a$ (log. scale) and 't' (natural scale) will produce a straight line of slope log k. Many more equations could be devised e.g., by adding another exponent 'n' to the ice melt exponential; but such equations do not appear in the literature.

A comparison of the shape of the various curves is shown in Fig. 3. As 't' increases, the exponential function with negative exponent becomes vanishingly small with far greater rapidity than the hyperbolic curve.

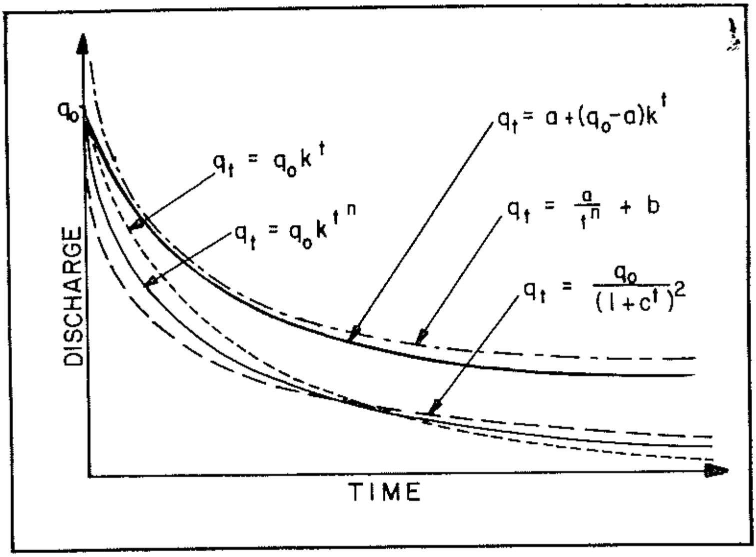

Fig. 3 — SHAPE of various recession curves.

MASTER RECESSION CURVES

Any storm hydrograph is a short-term event and its recession varies from the next one on account of variations in storage. Given a number of hydrographs of varying magnitude (thus covering a period of varying storage) they can be combined to give a master recession curve e.g., combining individual base flow recessions gives a master base flow recession curve.

Three methods are available for constructing master recession curves (i) correlation method,

(ii) strip method,

(iii) tabulating method.

The **correlation method** was suggested by Langbein (1940). It involves the plotting of q_t against q_{t+N} (q_t, N days later) on natural scales and drawing a straight line through the points and the origin; or alternatively plotting q_t against q_{t+N} on log-log paper and drawing a straight line through the points at 45^0 to the axes. The slope of this line is equivalent to log k and therefore the use of this method presupposes the fit of the simple exponential equation. Toebes and Morrissey (1962) suggested that if the plotted points indicated a curvilinear path, a curve should be drawn and they gave a method for subsequently deriving a master recession curve.

The **strip method** (Wisler and Brater, 1949; Toebes and Morrissey, 1962) involves plotting individual recessions on tracing paper; they are then superimposed and adjusted horizontally until the main parts overlap. A mean line through the overlapping parts is the master recession curve. This method is generally accurate because visual control allows omission of those parts of recessions which are too high (surface flow) or too low (snow melt). When the individual recessions are very flat it is difficult to decide where they fit together, and the resulting master recession curve may be either elongated or telescoped. This can be overcome by the use of a large magnification of the discharge ordinate.

The **tabulating method** as described by Johnson and Dils (1956) is essentially the same as the strip method. It involves the tabulation of daily mean discharges of individual recessions in columns (or instantaneous discharges at a fixed time, e.g., midnight). The columns are adjusted vertically until the discharges agree approximately horizontally. Subsequently the discharges are averaged horizontally and these mean discharges constitute the master recession curve. This method gives a good control of the data making it less probable that the final curve will be too long or too short. Its disadvantage is that irrelevant parts of the recession cannot be omitted without a detailed inspection.

To construct the master interflow recession curve the base flow effect must first be removed and to construct the surface recession the base flow and interflow effect must be removed. Methods of hydrograph analysis are numerous; detailed accounts are given by Linsley, *et al.* (1949), and Toebes (1962).

CURVE FITTING

Before fitting equations to recession curves the purpose of fitting them must be clearly known. For forecasting purposes an accurate fit especially at the lower end of the curve is most important. When making storage calculations an equation which fits the entire range of data is necessary. If, in particular, a simple functional relationship between storage and discharge, or storage and time is required, then, an equation which provides this simple relationship upon analytical integration is needed. In this case the simple exponential or hyperbola will be the most satisfactory. For catchment comparisons of recessions the simple exponential or the hyperbola are preferable since they have only one constant.

The allocation of a base flow recession equation to a catchment, simply from a theoretical consideration of whether the aquifers supplying base flow are confined or unconfined, is impossible in New Zealand at present, because the detailed geological structure of its catchments is not known. It is likely, in any case, that base flow from purely confined or unconfined aquifers is uncommon. Interflow and surface flow are outflows from unconfined storage; therefore from a theoretical point of view it is unlikely that the simple exponential will give the best agreement for any recession, or the hyperbola for base flow recession. Even without these considerations the double exponential is likely to give the best agreement since it has two parameters (a and n) as against only one for the simple exponential and hyperbola.

It may be argued that the ice melt exponential equation has two parameters also. However, the authors do not use this equation because the regression line derived from plots of q_t and q_{t+N} for Eq. (2) is of the form $q_t = c\, q_{t+N} - b$ which is difficult to imagine in practice since the resulting recession curve does not deplete to zero. The authors feel this only occurs in exceptional cases, for example in rivers strongly influenced by snow and ice melt. In the majority of cases a very flat curve through the origin gives a better fit to the plots of q_t and q_{t+N}. It is interesting to note that theoretically the plot of q_t and q_{t+N} for the double exponential results in a very flat curve of the form

$$q_t = q_{t+N} \cdot k^{(t^n - (t+N)^n)}$$

which passes through the origin. The hyperbola has the same property.

For a comparison of the simple, double and ice melt exponentials for the plot of q_t and q_{t+N} see Fig. 4.

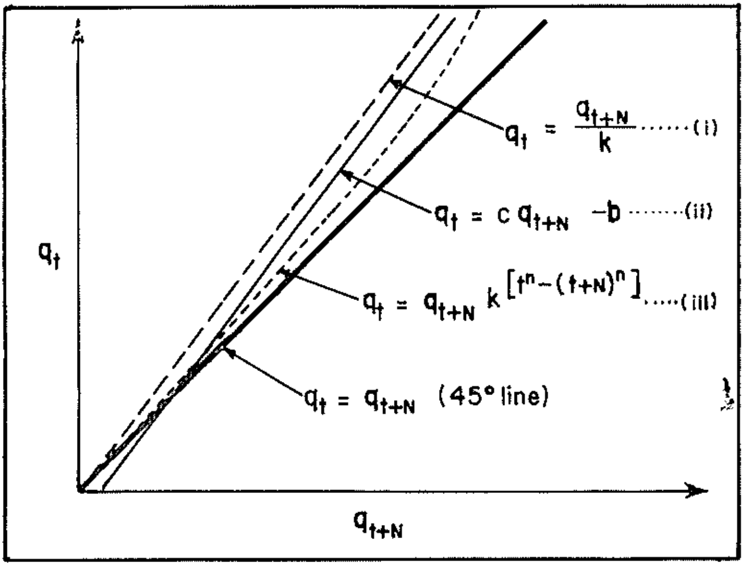

Fig. 4 — VARIOUS RECESSION CURVES derived by the correlation method for: (i) Simple exponential, (ii) Ice melt exponential, (iii) Double exponential.

For most basins so far investigated both the strip and tabulating method produce curvilinear master base flow and master interflow recession curves when plotted on semi-log paper (Anon., 1959, 1960), consequently the simple exponential equation cannot be fitted. The various equations were fitted to recession curves for some typical New Zealand catchments and the results are given below. As an indication of the accuracy of the fit the percentage deviations from the actual discharge at 10%, 40%, 70% and 100% of the total 't' of the observed recession curve are shown in Tables 1-7.

TABLE 1 — Master base flow recession for the Hutt River at Silverstream. (Vegetation — mainly native forest; rock type — greywacke; area — 216 sq. miles)

Observed Discharge (cusecs)	Time (days)	Percentage of Total Time	Percentage Deviation for Curve Fitted		
			Simple	Double	Hyperbola
201.5	1.4	10	—1.0	—1.2	—12.3
163.5	5.6	40	—3.5	+0.5	+3.3
141.2	9.8	70	+0.2	+2.1	+17.4
122.3	14.0	100	+4.1	+1.2	+26.7

TABLE 2 — Master base flow recession for the Manganui River at Tariki Road. (Vegetation — scrub, native forest; rock type — andesite; area — 31.5 sq. miles)

Observed Discharge (cusecs)	Time (days)	% of Total Time	Percentage Deviation for Curve Fitted: Simple	Double	Hyperbola
124.8 (170.0)*	2.1 (2.4)*	10	+5.9	—1.7	—12.8
78.8 (86.8)	8.4 (9.6)	40	—9.9	—5.9	—8.3
61.6 (63.4)	14.7 (16.8)	70	—3.6	—1.0	+7.2
54.0 (54.0)	21.0 (24.0)	100	+13.0	+10.2	+24.2

TABLE 3 — Master base flow recession for Makara No. 10 catchment. (Vegetation — grass; rock type — greywacke; area — 13.75 acres)

Observed Discharge (cusecs)	Time (days)	Percentage of Total Time	Percentage Deviation for Curve Fitted: Simple	Double	Hyperbola
0.0038	5	10	+2.6	+10.5	—27.0
0.0028	20	40	—3.6	—	—3.6
0.0023	35	70	—	—4.3	+14.3
0.0020	50	100	+10.0	—5.0	+30.0

TABLE 4 — Master base flow recession for Devil's Elbow catchment. (Vegetation — scrub and tussock; rock type — greywacke and schist; area — 0.55 sq. miles). This catchment is subject to frequent falls of snow and severe freezing in winter.

Observed Discharge (cusecs)	Time (days)	% of Total Time	Percentage Deviation for Curve Fitted: Simple	Double	Hyperbola	Ice Melt Hyperbola	Ice Melt Exponential
2.10	2.4	10	+17.1	—1.0	—18.1	+0.5	+10.5
0.59	9.6	40	—28.8	—18.6	—10.2	+13.6	—11.9
0.31	16.8	70	—6.4	—	+6.4	—3.2	—
0.21	24.0	100	+28.6	+23.8	+19.1	—14.3	+4.8

TABLE 5 — Master base flow recession for the Hooker River at Ball Hut Bridge (Land use — 60% permanent glaciers and snow, 35% bare rock, 5% scrub; rock type — greywacke and schist; area — 47 sq. miles). Hooker River catchment has seasonal recession curves and the figures were calculated for the summer-autumn period.

Observed Discharge (cusecs)	Time (days)	Percentage of Total Time	Percentage Deviation for Curve Fitted		
			Double Exponential	Ice Melt Hyperbola	Ice Melt Exponential
474	1.8	10	+4.0	—34.6	+3.3
222	7.2	40	—11.3	+18.9	+7.4
151	12.6	70	—4.8	—3.7	—1.6
151	18.0	100	+28.3	+1.0	+5.5

TABLE 6 — Master interflow recession for Makara No. 10 catchment. (Vegetation — grass; rock type — greywacke; area — 13.75 acres).

Observed Discharge (cusecs)	Time (hours)	Percentage of Total Time	Percentage Deviation for Curve Fitted		
			Simple	Double	Hyperbola
0.0592	8	10	+15.2	—4.2	—11.5
0.0211	32	40	—4.3	—	+5.2
0.0093	56	70	—4.3	+1.1	+5.4
0.0044	80	100	+1.1	+2.3	—8.0

TABLE 7 — Master interflow recession for Taita catchment. (Vegetation — regenerating native bush; rock type — greywacke; area — 37.8 acres).

Observed Discharge (cusecs)	Time (hours)	% of Total Time	Percentage Deviation for Curve Fitted			
			Simple	Double	Hyperbola	*Ice Melt Exponential (modified)
0.079	20.6	10	—0.6	—	—278.7	+3.8
0.038	82.4	40	+11.8	+15.0	+51.2	—2.6
0.016	144.2	70	+12.5	+15.6	+59.4	+1.0
0.002	195.7	95	—240.0	—280.0	—68.2	+55.0

* Since the master interflow recession curve was much less curved than the Makara interflow curve and depleted to zero after 206 hours, a modified ice melt exponential equation was fitted. Instead of 'a' being positive as in Eq. (5) it was made negative. However for 'a' negative the choice of 'a' becomes very arbitrary and the deviations shown here — although better than those for the other three curves fitted — apply only to one of many possible ice melt exponential equations.

TABLE 8 — Simple correlation coefficients

River	Simple	Double	Snow Melt Hyperbola	Snow Melt Exponential
Hutt	—0.87	0.98	—	—
Manganui	—0.92	0.99	—	—
Makara	—0.96	0.99	—	—
Devils Elbow	—0.92	0.98	—0.96	—0.98
Hooker	—	0.89	—	—

Some correlation coefficients for base flow curve fitting — corrected for the number of observations — are given in Table 8.

The tables show that for base flow recessions not strongly affected by snow and ice the double exponential gives the best fit. Some percentage deviations for the double exponential are still relatively high, e.g., + 10.2% at 100% of 't' for the Manganui; but this represents only 5.4 cusecs. It is felt that some master recessions have been drawn with a presupposed idea that the simple exponential should fit, thus making the end of the curve too flat.

For catchments strongly influenced by snow and ice it appears that either the ice melt hyperbola or the ice melt exponential can be used; but to date only two such catchments have been studied so that the superiority of one or other of the equations remains undecided.

Several equations have been fitted with some success to the interflow recessions, but the authors have found no general theory in the literature which favours one equation more than the other.

CONCLUSIONS

From this study of recessions it appears that equations other than the simple exponential give a good fit, especially the double exponential for base flow recessions. It is important to consider this when constructing master recession curves, so that no particular equation is presupposed and the construction adapted to that equation. The authors suggest that the reason why the simple exponential is still used so widely for smaller catchments (<500 sq. miles) is (i) simplicity, (ii) use of correlation method in establishing master recession curves, and (iii) lack of data. Individual recessions are normally too short to detect anything but a straight line when plotted on semi-log paper. While this is excellent for detecting breaks in the recession curve, it should not be assumed, because of an apparent fit of the simple exponential equation to short recessions, that the same equation fits the master recession.

RECOMMENDATIONS

The authors fully agree with Indri (1960) that "for practical purposes it seems better to set an empirical low water flow curve for each stream . . ." and they recommend that:

1. The ultimate use of the recession curve be defined before fitting an equation.

2. For forecasting purposes (extension of base flow recession curves) equations (1), (2) and (3) be tried in the case of catchments not strongly influenced by snow and ice.

3. For forecasting purposes all five equations be tried in the case of catchments strongly influenced by snow and ice.

4. For construction of master recession curves the strip or tabulating method be used: if the correlation method seems more satisfactory from a construction point of view then the simple exponential equation should not be presupposed: i.e., a straight line through origin of plot q_t versus q_{t+N}, or straight line at 45^0 through plot $\log q_t$ versus $\log q_{t+N}$.

   If recession curves or equations are used for catchment comparison only, then the correlation method seems entirely satisfactory.

5. More interflow and surface recession curves be derived. If the simple exponential, hyperbola and modified forms of these do not fit satisfactorily, then an equally simple equation be derived which fits the entire range of data and gives a simple storage function on analytical integration.

POSTSCRIPT

Just before this paper was transmitted to the editor, a publication by Riggs (1964) was received which describes the large influence of bank storage on recession curves. The authors agree with this for large catchments but the cases dealt with in their paper are all for small, steep catchments with relatively flashy streams. They agree entirely with Riggs' findings on the possibility of having more than one aquifer supplying base flow. This invariably results in complex base flow recession curves and underlines recommendation 5 of their paper.

ACKNOWLEDGMENTS

Publication of this paper is authorised by J. T. Gilkison, Commissioner of Works. The authors gratefully acknowledge computational assistance from Miss H. O'Driscoll.

REFERENCES

Anon., 1959: Base Flow Recession Curves. **Hydrol. Annual No. 7: 90.** S.C.R.C.C., Wellington.

——1960: **Hydrol. Annual No. 8:** 80. S.C.R.C.C., Wellington.

Barnes, B. S. 1939: The Structure of Discharge-Recession Curves. **Trans. Amer. Geophys. Un. Part IV;** 721-5.

Chapman, T. G. 1963; Effects of Ground Water Storage and Flow on the Water Balance. Nat. Symp. on Water Res. and Man. Canberra. C3. 12pp.

Horton, R. E. 1933: The Role of Infiltration in the Hydrologic Cycle. **Trans. Amer. Geophys. Un. 17th Ann. Meet.:** 448.

Indri, E. 1960; Low Water Flow Curves for some Streams in the Venetian Alps. **I.A.W.H., Gen. Ass. of Helsinki. Publ. No. 51:** 124-9,

Johnson, E. A.; Dils, R. E. 1956; Outline for Compiling Precipitation, Runoff, and Ground Water Data from Small Watersheds. **U.S. Dept. Agr., Forest Service Tech. Note No. 34.**

Langbein, W. B. 1940; Some Channel Storage and Unit Hydrograph Studies, **Trans. A.F.U. 21:** 620-7.

Linsley, R. K.; Kohler, M.A.; Paulhus, T. L. H. 1949; **Applied Hydrology.** McGraw Hill, New York.

Maillet, E. 1905: **Essais d'hydraulique,** Paris.

Riggs, H. C. 1964: The Base-Flow Recession Curve as an Indicator of Ground Water. **Intern. Assoc. Sci. Hydrol., Gen. Ass. of Berkeley, Publ. No. 63:** 352-63.

Toebes, C.; Morrissey, W. B. 1962: Base Flow Recession Curves. **Handbk. of Hydr. Proc., Prov. Proc. No. 8** S.C.R.C.C., Wellington

Toebes, C. 1962: Hydrograph Analysis. **Hydrology and Land Management, Proc. of Tech. Sympos.:** 49-65. S.C.R.C.C., Wellington.

Werner, P.W.; Sundquist, K. J. 1951: On the Ground Water Recession Curve for Large Watersheds. **Intern. Assoc. Sci. Hydrol.. Gen. Ass. of Brussels, Publ. No. 33:** 202-12.

Wicht, C. L. 1943: Determination of the Effects of Watershed Management on Mountain Streams. **Trans. Amer. Geophys. Un. Part II:** 594-605.

Wisler, C. O.; Brater, E. F. 1949: **Hydrology.** Chapman and Hall, London.

Comments on this paper should be forwarded to the Editor before 1 April. 1965.