CALCULATION OF THE INSTANTANEOUS UNIT HYDROGRAPH USING LAPLACE TRANSFORMS

P. Johnson*

ABSTRACT

Most of the recently derived methods for the calculation of the instantaneous unit hydrograph using observed rainfall-runoff data from a catchment are suggested to have in common the use of integral transform equations and linkage equations. A Laplace transform technique is demonstrated as having direct analogy with the method of moments. Potential accuracy of the Laplace method is illustrated by analysing synthetic data and number of discharge data before the peak discharge, and smallness of the transform parameter, are proved to be important criteria. Analyses of real data indicate the difficulty in applying such a method in practice until better understanding of data inaccuracy and hydrograph models is obtained.

INTEGRAL TRANSFORMS AND DERIVATION OF THE INSTANTANEOUS UNIT HYDROGRAPH

Calculation of a storm runoff hydrograph by convoluting effective rainfall with a unit hydrograph (U.H.) or an instantaneous unit hydrograph (I.U.H.), is a criticised but nevertheless accepted procedure in hydrology. It is represented by the integral equation:

$$Q(t) = \int_0^t i(\tau) \cdot u(0, t-\tau)\,\mathrm{d}\tau \qquad (1)$$

where $Q(t)$ is the storm discharge hydrograph,
$i(\tau)$ is the effective intensity of rain,
$u(0, t-\tau)$ is the instantaneous unit hydrograph (I.U.H.).

The determination of the I.U.H. using observed rainfall and runoff data in association with the convolution integral is also a well recognized and much discussed problem. Active research has gone into solving this problem and as a result, hydrologists are now aware of a number of possible techniques by which the I.U.H. or the U.H. may be established.

The oldest and perhaps best known and most used method, even today, attempts to calculate successive ordinates of the U.H.

* Department of Civil Engineering, University of Newcastle upon Tyne, England.

by successive solution of a set of equations representing consecutive ordinates of the storm runoff hydrograph. The technique is not to be recommended, however, as any small error present in initial computations tends to be amplified in succeeding calculations. To overcome such inaccuracy, a matrix method using a least-squares fit may be used to solve the same set of equations (Snyder, 1955). The least-squares method has a proven accuracy, a fact recently reinforced by Laurenson and O'Donnell (1969). Since its inception certain modifications and refinements have been suggested, one notable example being the linear programming solutions of the Wiener-Hopf equation with rational restraints (Eagleson *et al.*, 1966).

Alternative procedures to the least-squares method have been studied for the last 15 years. During that time two basic techniques have emerged.

The first of these is related almost entirely to the mathematical subject of integral transforms which have the following general form:

$$T(s) = \int_a^b \mathrm{f}(t) \cdot \varphi(s,t)\,\mathrm{d}t \qquad (2)$$

where $\mathrm{f}(t)$ is a known numerical or algebraic function of the independent variable t,

$\varphi(s,t)$ is a known function of the independent variable t and a parametric variable s,

a, b are the limits of integration for t,

$T(s)$ the transform is a resulting function in s only.

In simple terms $T(s)$, defined numerically or otherwise, may be looked upon as representing areas bounded by two limits of t and parametric curves in s resulting from the product of $\mathrm{f}(t)$ with $\varphi(s,t)$. It happens that knowledge of $T(s)$ as a function of s may be used to determine useful properties of $\mathrm{f}(t)$ and consequently of the various functions involved in convolution.

An illustration of the use of direct transforms in convolution is given by Nash (1957, 1960). The transform adopted is a moment transform, and the method developed to determine the I.U.H. is referred to as the method of moments. The general transform is given by

$$M(s) = \int_0^\infty \mathrm{f}(t) \quad (t - \bar{t})^s \mathrm{d}t \qquad (3)$$

where $\bar{t}$ is the interval in t from the origin to the centre of area of $\mathrm{f}(t)$.

The result of applying this transformation to the runoff hydrograph and the convolution integral is a simple moment-transform generating function which performs as a linkage equation between the moments of rainfall, runoff and the I.U.H.

$$Q_s = (I + U)^s \tag{4}$$

In this equation I^s and U^s in the expansion of the right-hand side represent the sth order moments of I and U respectively (I_s and U_s), not the sth power of the 1st moments. Q_s is the sth moment transform of $Q(t)$.

Numerical computation of I^s and Q^s etc., permits the direct calculation of the corresponding U_s. Nash shows how, in calculating only the first two moments ($s=1$ and $s=2$), appropriate parameter values of a two-parameter gamma distribution model of the I.U.H. may be determined. In this way he effects the all-important inversion of the transform.

A better known transform is the Laplace

$$L(s) = \int_0^\infty \mathrm{f}(t) \quad e^{-st} \, dt \tag{5}$$

This transform has special relevance in the field of systems analysis and its main features are well documented. It is especially significant in that it reduces the convolution integral into a simple equation from which the Laplace transform of the I.U.H. is easily calculable (Brown, 1965).

i.e.
$$\overline{U}_s = \overline{Q}_s / \overline{I}_s \tag{6}$$

where $\overline{Q}_s$ is the Laplace transform of $Q(t)$
$\overline{I}_s$ is the Laplace transform of $i(\tau)$
$\overline{U}_s$ is the Laplace transform of $u(t)$

This equation is also a linkage equation.

Determining $u(t)$, the inverse of $\overline{U}_s$, is a problem which, although referred to by several authors, has not yet been successfully achieved directly. In analogous fields of research this is not the case and accurate general inversion does seem possible through the use of Legendre polynomials and a quadrature technique (Bellman *et al.*, 1966).

In hydrology it seems that the only major work using Laplace transforms was performed by Diskin (1967, 1968), who made a detailed study of the transfer function (the Laplace transform of the I.U.H.) and a special transfer function in relation to a number of mathematical models of the I.U.H. He also investigated the dependence of the special transfer function on different storms and different catchments. Diskin (1967), O'Connor and Nash (1968), and Diskin and Boneh (1968), have all discussed at length the relationship between the method of moments and Laplace transforms.

A second alternative procedure to least squares also uses integral transforms, but in this method the transforms are used only to calculate coefficients of series. Two notable examples of the series method are O'Donnell (1961) and Dooge (1964). The technique begins by assuming a function $\mathrm{f}(t)$ to be represented by an orthogonal series:

$$\mathrm{f}(t)=\sum_{p=0}^{\infty} A_p \ \theta(p,t) \tag{7}$$

where A_p is the general coefficient in the series, to be determined; $\theta(p,t)$ is a function of the independent variable t and the parameter p.

The orthogonality property of the series permits the calculation of each coefficient in the following manner:

$$A_p=w(p)\int_a^b \mathrm{f}(t) \ . \ \theta(p,t)\mathrm{d}t \tag{8}$$

where $w(p)$ is a necessary 'weighting' function in p; and it is noted that in this sense the coefficients are integral transforms. In practice the integral is closely approximated by numerical calculation of an equivalent finite series.

Substitution in the convolution integral of three such series, representing rainfall, runoff and the I.U.H. results, by identity, in linkage equations between the three sets of coefficients. By first calculating the series coefficients of runoff and rainfall using a series approximation to the integral transforms, the corresponding unknown coefficients of the I.U.H. may then be determined. The linkage analogy between this technique and the direct transform technique as demonstrated by Nash is quite clear, but the series method has the immediate advantage that inverse transforms are not required.

LAPLACE TRANSFORM AND THE GAMMA MODEL

The general implications of the foregoing brief survey of methods to determine the I.U.H. indicates that any direct transform method or orthogonal series method can be expected to reduce to the solution of intergral transforms which, when combined with the convolution integral, result in linkage equations. For the direct transform method only, an inverse transform will be required to define the I.U.H. as a function of time. This may be achieved in one of two ways: firstly and approximately by assuming the I.U.H. to be of a definite mathematical form, or secondly and more accurately by using series approximations in conjunction with quadrature techniques. The former method has recently been studied by the author with reference to the Laplace transform.

The method is illustrated by assuming the I.U.H. to be represented by the gamma distribution, after Nash (1957).

$$u(0,t) = (1/k\sqrt{n})\,(t/k)^{n-1}\ e^{-t/k} \qquad (9)$$

where n,k are two parameters, suitable values of which are to be determined.

The Laplace transform of the above expression is well known:

$$\overline{U}_s = 1/(1+ks)^n \qquad (10)$$

Assuming that the Laplace transforms of discharge and rainfall can be calculated for at least two values of s, g and r respectively, then through linkage equation (6) we obtain

$$\left.\begin{aligned}(1+kg)^n &= \overline{I}_g/\overline{Q}_g\\ (1+kg)^n &= \overline{I}_r/\overline{Q}_r\end{aligned}\right\} \qquad (11)$$

Equations (11) may be combined and simplified to give

$$(1+rk) = (1+gk)^z \qquad (12)$$

where

$$z = \frac{\ln(\overline{I}_r/\overline{Q}_r)}{\ln(\overline{I}_q/\overline{Q}_q)} \qquad (13)$$

Equation (12) is implicit in only one unknown k. Solving for k, n may be calculated directly from

$$n = \frac{\ln(\overline{I}_q/\overline{Q}_q)}{\ln(1+kg)} \qquad (14)$$

The method implied in the above equations to determine n and k is analogous to the method of moments; there are three steps in the solution:

(a) Integral transforms of rainfall and runoff data are determined by equation (5) (usually numerically).

(b) Corresponding transform values of the I.U.H. are then determined through the linkage equation (6).

(c) Inverse transformation is performed through equations (12) to (14), which are thereby equivalent to Nash's equations (4) (1957).

In contrasting this technique with the method of moments it is readily appreciated that in a hand computation the method of moments is the easier to apply. This is especially true in the calculation of k, for which the Laplace transform method involves the solution of an implicit equation. When programmed for computer evaluation, however, the length and complexity of computation in either method ceases to be of significance. Also, solution of equation (12) may be achieved quite simply. An initial estimate of k may be determined from the equation

$$k=3/g(r/gz)^{1/(z-1)} \tag{15}$$

after which only one or two closer approximations, using the Newton-Raphson iteration process, are required to evaluate k to within one percent of its true value.

POTENTIAL ACCURACY OF THE METHOD

The general accuracy of the proposed method is considered to depend on at least three possible sources of inaccuracy. These may be summarized as follows:

(a) Basic inaccuracy in rainfall-runoff data due to errors in measurement, estimation of areal rainfall, separation of effective components and synchronization of data measurement.

(b) Failure to represent accurately rainfall and runoff data, resulting in erroneous estimates of the Laplace transforms.

(c) A general instability involved in the numerical inversion of the I.U.H. transform during the calculation of n and k.

The first of these may possibly be considered the most important problem of the three but is outside the intended scope of this paper. The many attendant problems associated with (a), however, certainly cannot be investigated with assurance until sources of inaccuracy represented by (b) and (c) are investigated and obviated.

Rainfall-Runoff Representation and Transforms

Standard representation of rainfall is a simple bar diagram, each bar representing an average intensity of rainfall during a period of time of suitably small duration. In an overdamped system, such as a natural catchment, this type of representation is certainly suitable provided the intensity averages are for not too long a duration and fairly represent the catchment areal average. The technique is well illustrated by Diskin (1964) and is summarized here:

$$i(t) = I_1[H(t) - H(t-\tau)] + I_2[H(t-\tau) - H(t-2\tau)] + \ldots\ldots$$
$$\ldots\ldots + I_m\{H[t-(m-1)\tau] - H(t-m\tau)\} \qquad (16)$$

where I_1, I_2 . . . . I_m are the average effective rainfall intensities during time intervals of duration τ; $H(t)$, $H(t-\tau)$ etc. are the Heaviside unit function; m refers to the last discrete duration of effective rain. The Laplace transform of this equation is of a standard form:

$$\overline{I}_s = [I_1 + (I_2 - I_1)e^{-s} + (I_3 - I_2)e^{-2} + \ldots\ .$$
$$\ldots\ldots + (I_m - I_{m-1})e^{-(m-1)s} - I_m e^{-ms}] \qquad (17)$$

Runoff representation is not as straightforward. Early attempts by the author at calculating Laplace transforms of runoff were based on forming individual products between $Q(t)$ and e^{-st} for increasing values of t and then forming the sum of incremental areas between adjacent product pairs. These attempts were abortive and led to wildly inaccurate estimates of k and n. A mathematical representation was sought therefore which would not only interpolate values of runoff accurately but which would also permit simple determination of the Laplace transform. The most suitable function studied was the Fourier sine series.

$$Q(t) = \sum_{p=1}^{m} b_p \sin\ (pwt) \qquad (18)$$

The Laplace transform of this series is given by

$$\overline{Q}_s = \sum_{p=1}^{m} \frac{b_p sw}{[s^2 + (pw)^2]}(1 - e^{-sT}) \qquad (19)$$

where m is the number of data values,

b_p is the general coefficient of the series,

T is the total duration of the discharge hydrograph,

w is the basic harmonic.

The method seems to have three major attributes. Firstly, it provides good curvilinear interpolation between data points and hence helps to ensure accurate transforms. Secondly, it gives absolute fit at the origin, a feature which is not true of the full Fourier series; and as the Laplace transform is notably sensitive to large errors for small t, this criterion is of paramount importance. Thirdly, it seems to avoid negative values of transforms for large s due to small inaccuracies of fit, a feature which was occasionally found to occur in the full Fourier series.

The terms in parentheses in the numerator of the algebraic expression of the series, equation (18), are introduced to reduce the transform of the periodic function over the entire interval $0<t<\infty$, to one only for the interval $0\leqslant t\leqslant T$. Usually this term for all practical purposes may be assumed to be unity.

Accuracy of fit by the Fourier sine series was generally found to be within 10^{-8} parts of the basic unit of the data values.

Instability of *k* and *n* Calculations

To study basic instability, a computer programme was written embodying the general principles of the method outlined. Added to the programme was a procedure to synthesize a set of discharge data for a given pair of *n* and *k* values and defined patterns of effective rainfall. The programme was designed to analyse the synthetic data and calculate a series of values of *n* and *k* corresponding to a series of paired values of *g* and *r* as shown in equations (11).

Two synthetic procedures were adopted. The first and simplest assumed an instantaneous input of rainfall. In this way numerical evaluation of the unit hydrograph from the I.U.H. and any associated inaccuracies were avoided. The only inaccuracies possible were those due to rounding off within the computer, small errors in the lack of fit of discharge data by a Fourier sine series, and instability of the equations being analysed in determining the inverse of the transform. Three pairs of *n* and *k* values were studied, representing an expected practical range. In conjunction with each pair, the effect of time interval size between runoff data values and hence number of runoff data was also investigated.

The second synthetic procedure allowed for simple discrete inputs and combinations of discrete inputs of different magnitude, equations (16) and (17). In this, numerical evaluation of the unit hydrograph was necessary.

Results of instantaneous input investigations are shown in graphical form (Figs. 1, 2 and 3). These show that calculations of *n* and *k* by the method described, for different values of the parameter pairs *g* and *r*, results in a field of contours representing different values of *n* and *k*. The general pattern of behaviour of the solution would seem to depend on the value of *n* and *k* used to synthesize runoff data; this implies that the technique is sensitive to the peakedness of the discharge hydrograph.

Correct values of *n* and *k* are achieved, and these are seen to occur for small values of *g* and *r*. Values of *g* and *r* to give the correct value of *k* are not necessarily the same as for the correct value of *n*.

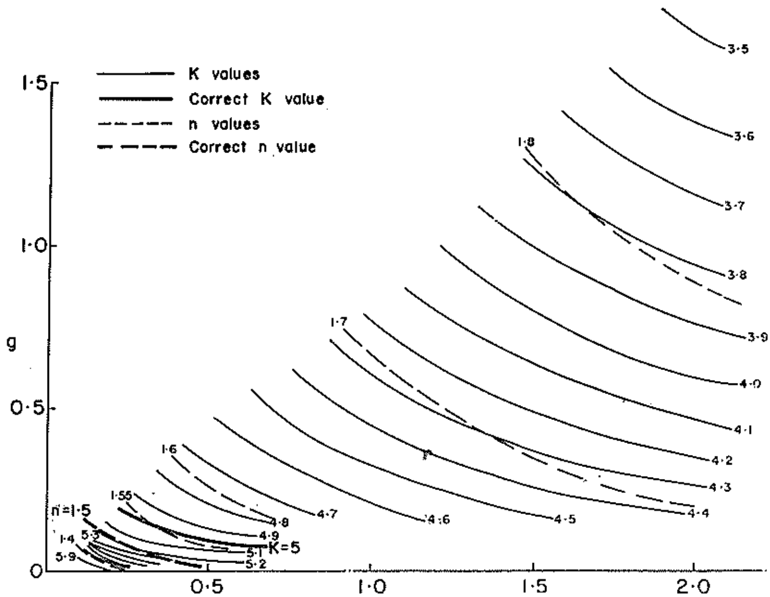

FIG. 1 — Solution contours for $n=1.5$, $k=5.0$.

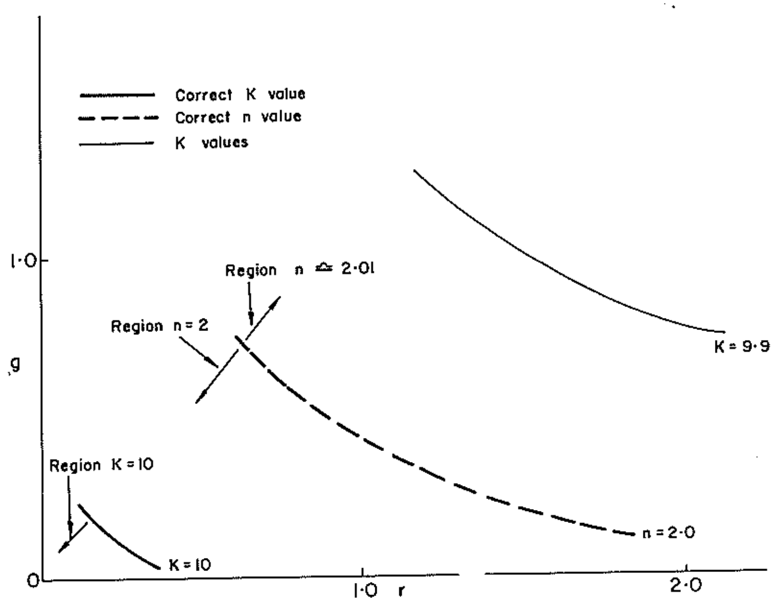

FIG. 2 — Solution contours for $n=2$, $k=10$.

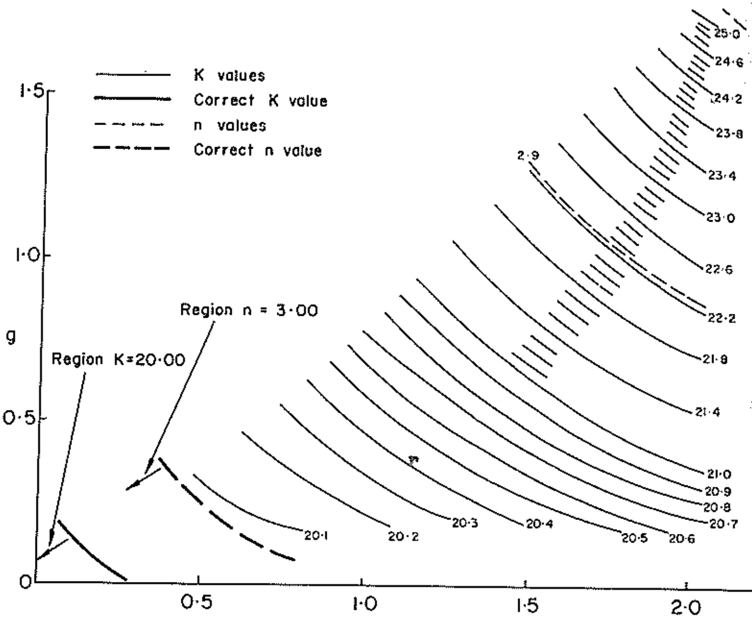

FIG. 3 — Solution contours for $n=3$, $k=20$.

Results for rainfall input of finite duration having varying distribution of intensity with time are found to behave in exactly the same way, indicating that numerical evaluation of the integral to derive the unit hydrograph for convolution had negligible effect on the process. A summary of the main features of the investigation is given in Table 1.

Investigation of the results led to two major conclusions:

(a) Using a Fourier sine series to represent a discharge hydrograph, the possibility of obtaining correct values of n and k will only be assured if the peak of the hydrograph occurs at least 4, and preferably 6 to 10, time intervals after the start of the hydrograph. The absolute value of the time interval and number of data used seem incidental to this requirement. The rapid divergence of n and k values shown in Figure 1 for small values of g and r is considered to be entirely due to too few data before the peak discharge and hence lack of curvature fit of the Fourier sine series.

(b) Even with the above criterion satisfied, values of n and k are still noted to deviate from their true values as g and r increase. As this could have been due to small differences between the actual and fitted hydrographs, a small analysis was carried out to study the effect. (It is mentioned in passing that the same analysis would

TABLE 1 — Summary of tests on the accuracy of calculated n and k values.

Number of discharge data	Rainfall Type of input	Data time interval (hours)	Parameter values used in synthetic data		Approximate number of time intervals to peak of discharge hydrograph	General values of s (r and g) for which calculated and are acceptable*
			n	k (hours)		
180	instantaneous	0 (Rainfall) 1 (Discharge)	1.5	5	3	Unacceptable except for isolated values (A line on Fig.1)
			2	10	10	$s \leqslant 2.0$ (Further values of s not tested - Fig.2)
			3	20	40	$s \leqslant 0.6$ (Fig.3)
180	discrete duration	1	1.5	5	3	Unacceptable except for isolated values (Similar to Fig.1)
			2	5	6	$s \leqslant 2.0$ (Further values of s not tested) (Similar to Fig.2)
			2	10	11	$s \leqslant 0.45$
			3	20	NOT	ANALYSED
90	discrete duration	2	1.5	5	2	Unacceptable
			2	10	6	$s \leqslant 0.25$
			3	20	21	$s \leqslant 0.60$
60	discrete duration	3	1.5	5	2-3	Unacceptable
			2	10	4	$s \leqslant 0.15$
			3	20	14	$s \leqslant 0.35$
30	discrete duration	6	1.5	5	1	Unacceptable
			2	10	3	$s \leqslant 0.15$
			3	20	7	$s \leqslant 0.20$

* Acceptable values are taken to mean that k is within 0.2 of its true value and n is within 0.05 of its true value.

be very useful in studying also the effects of inaccurate separation of stormwater from base flow, during the analysis of real data.) An equation was developed which enabled an estimate of error in k due to errors in the discharge hydrograph:

$$\Delta k = (2/g)(r/gz)^{1/(z-1)} \frac{[(\Delta\overline{Q}_g/\overline{Q}_g)z - (\Delta\overline{Q}_r/\overline{Q}_r)]\,[\ln z - \ln(r/g) - (z-1)/z]}{(z-1)^2 \ln(\overline{I}_g/\overline{Q}_g)} \qquad (20)$$

where Δk is the expected error in k,

$\Delta\overline{Q}_g$ and $\Delta\overline{Q}_r$ are the calculated errors of Laplace transforms of discharge data due to an error in the data, $\Delta Q(t)$.

In a computer programme the equation is easily evaluated and was thereby used to estimate expected error in k due to observed lack of fit. Values of Δk were generally found to be of the same order of magnitude as (sometimes greater and sometimes less than) the order of magnitude of the lack of fit.

It was concluded therefore that the general divergence of k and n values as g and r increase, is probably due to sensitivity of the technique to numerical evaluation. To guarantee any chance of obtaining appropriate values of n and k for real data, therefore, only values of g and r in the region of 0.05 to 0.2 should be used.

TRIALS WITH REAL DATA

This preliminary study into the possibility of obtaining an accurate inverse of the Laplace transform of the I.U.H. was completed with the analyses of real flood hydrograph and rainfall data from different catchments. For reasons already referred to and

TABLE 2 — Variation in k and n values derived from real data for Ashbrook, England.

	Parameters		
g	r	k	n
0.05	0.10	16.08	1.49
	0.15	15.12	1.56
	0.20	14.42	1.62
	0.25	13.84	1.67
	0.30	13.30	1.72
	0.35	12.81	1.77
	0.40	12.35	1.83
	0.45	11.92	1.88
	0.50	11.52	1.93

TABLE 2 — (contd.)

	Parameters		
g	*r*	*k*	*n*
0.10	0.15	13.65	1.66
	0.20	12.98	1.72
	0.25	12.38	1.77
	0.30	11.83	1.83
	0.35	11.31	1.89
	0.40	10.83	1.94
	0.45	10.39	2.00
	0.50	9.99	2.06
0.15	0.20	12.07	1.79
	0.25	11.43	1.85
	0.30	10.85	1.91
	0.35	10.31	1.98
	0.40	9.81	2.04
	0.45	9.36	2.11
	0.50	8.95	2.17
0.20	0.25	10.65	1.92
	0.30	10.04	1.99
	0.35	9.48	2.06
	0.40	8.97	2.14
	0.45	8.52	2.21
	0.50	8.11	2.28
0.25	0.30	9.33	2.08
	0.35	8.76	2.15
	0.40	8.25	2.23
	0.45	7.80	2.31
	0.50	7.41	2.38
0.30	0.35	8.12	2.24
	0.40	7.62	2.33
	0.45	7.19	2.41
	0.50	6.81	2.49
0.35	0.40	7.08	2.42
	0.45	6.66	2.51
	0.50	6.30	2.59
0.40	0.45	6.21	2.61
	0.50	5.87	2.69
0.45	0.50	5.52	2.79

the simple fact that the gamma model is probably not fully representative of all possible I.U.H. shapes, the analyses of the real data were not expected to yield consistent results, nor did they. Because of a useful direct comparison with values derived by Nash (1957), only the results for data from Ashbrook, England, will be referred to here.

TABLE 3 — Example of results using real data for Ashbrook, England. (n=1.83, k=11.83)

Time (hrs)	*I.U.H. (cusecs)*	*Predicted discharge (cusecs)*	*Actual discharge (cusecs)*
0.00	0.000	0.00	0.00
3.00	13.996	73.53	30.00
6.00	19.729	291.44	340.00
9.00	21.587	822.01	980.00
12.00	21.344	1249.72	1320.00
15.00	19.975	1391.30	1390.00
18.00	18.059	1386.23	1280.00
21.00	15.942	1302.90	1160.00
24.00	13.832	1181.13	1040.00
27.00	11.843	1044.69	910.00
30.00	10.034	907.65	790.00
33.00	8.431	777.96	680.00
36.00	7.035	659.73	580.00
39.00	5.836	554.69	480.00
42.00	4.817	463.09	390.00
45.00	3.959	384.33	320.00
48.00	3.242	317.36	280.00
51.00	2.646	260.93	240.00
54.00	2.153	213.73	210.00
57.00	1.748	174.49	180.00
60.00	1.416	142.03	155.00
63.00	1.144	115.32	135.00
66.00	0.923	93.41	115.00
69.00	0.743	75.50	100.00
72.00	0.597	60.91	85.00
75.00	0.480	49.05	70.00
78.00	0.384	39.44	65.00
81.00	0.308	31.67	60.00
84.00	0.246	25.39	55.00
87.00	0.197	20.34	50.00
90.00	0.157	16.27	45.00
93.00	0.125	13.00	40.00
96.00	0.100	10.38	35.00
99.00	0.079	8.27	30.00
102.00	0.063	6.59	25.00
105.00	0.050	5.25	15.00
108.00	0.040	4.18	5.00
111.00	0.032	3.32	0.00

The range of n and k values calculated for various pairs of g and r is shown in Table 2. The diversity, particularly for small g and r, is obviously the result of using either an inaccurate model or inaccurate data, or both. Convolution of the I.U.H. with effective rainfall was performed for each n and k pair, and one example of the result is compared with the actual discharge hydrograph (Table 3). This example, like several others, is not very satisfactory and is neither better nor worse than the result derived by Nash for n and k values of 2.1 and 10.8 hours respectively. As an illustration, however, it does point to the possible applicability of such a method when there is perhaps more understanding of data inaccuracy and separation techniques, and a more generally representative I.U.H. model.

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REFERENCES

Bellman, R.; Kalaba, R. E.; Lockett, J. A. 1966: *Numerical Inversion of the Laplace Transform*. Elsevier, New York.

Brown, B. M. 1965: *The Mathematical Theory of Linear Systems*. Chapman and Hall, London.

Diskin, M. H. 1964: *A Basic Study of the Linearity of the Rainfall-Runoff Process on Watersheds*. Ph.D. thesis, University of Illinois.

Diskin, M. H. 1967: A Laplace transform proof of the theories of moments for the instantaneous unit hydrograph. *Wat. Resources Research 3*: 385–388.

Diskin, M. H. 1968: Transfer functions for the analysis of rainfall-runoff relations. In: *Int. Ass. Sci. Hydrol. Publ. No. 85.*

Diskin, M. H.; Boneh, A. 1968: Moments of input, output, and impulse response functions of linear systems about arbitrary points. *Wat Resources Research 4* (4) : 727–735.

Dooge, J. C. I. 1964: Analysis of linear systems by means of Laguerre functions. *Jour. Soc. Ind. App. Math. 2* (3) : 396–408.

Eagleson, P. S.; Mejia-r, R.; March, F. 1966: Computations of optimum realizable unit hydrographs. *Wat. Resources Research 2* (4) : 755–764.

Laurenson, E. M.; O'Donnell, T. 1969: Data error effects in unit hydrograph derivation. *Proc. A.S.C.E. 95* (HY6) : 1899–1917.

Nash, J. E. 1957: The form of the instantaneous unit hydrograph. In: General Assembly of Toronto, vol. 3. *Int. Ass. Sci. Hydrol. Publ. No. 45.* pp. 114–121.

Nash, J. E. 1960: A unit hydrograph study with particular reference to British catchments. *Proc. I.C.E. 17*: 249–282.

O'Connor, K. M.; Nash, J. E. 1968: Comment on "A Laplace proof of the theorem of moments" by M. H. Diskin. *Wat. Resources Research 4* (3) : 675–677. (see also pp. 679–680.)

O'Donnell, T. 1961: Instantaneous unit hydrograph derivation by harmonic analysis. *In*: General Assembly of Helsinki. *Int. Ass. Sci. Hydrol. Publ. No. 51.* pp. 546–557.

Snyder, W. M. 1955: Hydrograph analysis by the method of least squares. *Proc. A.S.C.E. 793.*