NOTE

Transit time distributions are not L-shaped

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Abstract

A probability density function $f(t)$ with origin at $t = 0$ is defined here as being 'L-shaped'if $f'(t) \leq 0$ for $t \geq 0$. L-shaped probability density functions, such as exponential distributions, are often employed as transit time distributions in hydrological modelling. However, the use of L-shaped transit time distributions implies including tracer particles that were already present at the observation point at time $t = 0$. These passive particles have no transit history by definition because 'transit' implies some history of movement of a particle through a hydrological system, however small that movement may be. That is, the particle must have reached the observation point by moving to it for some non-zero time and over some non-zero distance through a hydrological system such as a catchment or aquifer. By this argument, particles that happen to be already at the observation point at time $t = 0$ represent background noise.

The distinction between the inclusion or not of particles already at the observation point might seem pedantic but does have an important implication: if transit time distributions in the hydrological environment are defined to not include non-transiting particles, they must have the property $f(0) = 0$. This negates the possibility of L-shaped transit time distributions. The exclusion of non-transiting particles also has the implication that transit time distributions must have at least one mode at $t > 0$, which may or may not be identifiable from field measurements.

Keywords

Introduction

A hydrological transit time distribution, $f(t)$, arises as the distribution of travel times of a set of tracer particles introduced by artificial or natural processes into a hydrological system at defined time $t = 0$, with individual particle exit times being subsequently recorded at an observation point. This brief paper is concerned with making the case that L-shaped probability density functions have forms that are not strictly compatible with the concept of transiting particles in hydrology. A probability density function $f(t)$ with origin at $t = 0$ is defined here as being 'L-shaped' if $f'(t) \leq 0$ for $t \geq 0$. This includes exponential distributions and gamma distributions with shape parameters < 1.

In principle, exponential transit time distributions might arise if it were possible to have a well-mixed situation where tracer particles within the system all had exactly equal probability of departing via the observation point in a small time interval, irrespective of their spatial location in the

system. This is clearly impossible because there will be unavoidable correlations between particle locations and exit probabilities.

As noted by Leray *et al.* (2016), derivations of exponential transit time distributions have also been proposed in non-mixing contexts for some idealised aquifer situations (Eriksson, 1958; Haitjema, 1995; Leray *et al.*, 2012; Raats, 1977; Vogel, 1967). In addition to exponential distributions, L-shaped gamma transit time distributions have been used in a number of catchment studies (Kirchner *et al.*, 2000; Kirchner *et al.*, 2010; Hrachowitz *et al.*, 2010; Godsey *et al.*, 2010). Fiori and Russo (2008) reported computer-simulated transit times similar to an L-shaped gamma distribution. A conceptual model of water moving downslope to a river channel was utilised by Kirchner *et al.* (2001) to derive an L-shaped transit time distribution as an infinite mixture of inverse Gaussian distributions.

However, L-shaped transit time distributions require a definition that some non-transiting passive particles also be included. Specifically, tracer particles that happen to be already at the observation point at time $t = 0$ are implicitly defined to be part of the transit time distribution of the hydrological system concerned. Such particles would all have transit times of zero, which itself is something of a contradiction of terms.

To argue against L-shaped distributions on such definitional grounds may seem pedantic and obviously cannot be verified by field measurements, now or at any time in future. The issue is nonetheless worth considering because L-shaped distributions under this definition will give an inaccurate representation of transit time probability density functions near $t = 0$. Maintaining the definition of including non-transiting particles leads to some logical inconsistencies, as illustrated in the following section.

Illustrations

Firstly, there is the problem of attempting a qualitative hydrological description of L-shaped transit time distributions. For example, exponential transit time distributions have been characterised somewhat awkwardly as including transit times that are 'very short' (McGuire and McDonnell, 2006) or 'infinitesimal short' (Amin and Campana, 1996). Secondly, there is a philosophical issue of having to group together two very different populations of transiting particles – transiting particles that have transited and 'transiting' particles that actually have no record of transit. Defining these two histories into a single population is like having a mixture of apples and oranges and then calling them all oranges.

Thirdly, there is a logical inconsistency that can be illustrated by the following thought experiment, being an idealised catchment tracer experiment with a perfect recorder detecting every tracer particle departing within the stream discharge at a point at the lower end of the catchment. Every tracer particle that enters the catchment is deemed to leave sooner or later via the recorder site.

Suppose in an instant of time at $t = 0$, raindrops containing tracer particles fall over the extent of the catchment, with one raindrop falling exactly on the recorder. Denote M to be the number of tracer particles in the recorder raindrop and N is the total number of tracer particles deposited by the other raindrops in the catchment upstream of the recorder. These N tracer particles eventually all depart the catchment via stream discharge at varying times $t_1, t_2 \ldots t_N$.

On the basis of this idealised experiment, it is possible to write out two different expressions for mean transit time, μ:

$$\mu = \sum_{i=1}^{N} t_i / N \qquad (1)$$

$$\mu = \sum_{i=1}^{N} t_i / (N + M) \qquad (2)$$

Equation 1 is based on the definition of transiting particles being only those that have reached the recorder as a consequence of movement to it. Equation 2, on the other hand, extends the definition of transiting particles to include those M particles already at the recorder at $t = 0$, which have no history of movement through the catchment system. If Equation 2 is correct then there would be the strange situation of mean transit time being determined in part by a raindrop that never entered the catchment.

Finally, it is recalled that all hydrological transit time distributions are first passage times, with transits being terminated at the recorder site and never recorded a second time. In other words, the recording site serves as an absorbing barrier. For example, the topographic model of Kirchner *et al.* (2001) considers downslope transits of water particles to a river channel (absorbing barrier), with the ridge crest effectively being a reflecting barrier.

The issue now is that if transiting particles are defined to be permissibly 'at' the recording site at time $t = 0$, then this equates to permitting an absorbing barrier to be the source of transits to itself. There is a logical difficulty here because if particles vanish in the same instant they are created then the particles never existed in the first place and there can be no transit times of zero. For this reason, when considering the mathematical specification of a random walk starting at $t = 0$ at some point x between a reflecting barrier R and an absorbing barrier A, the initiating point is bounded as $A < x \leq R$ (Weesakul, 1961). That is, no particle has yet reached the absorbing barrier A at $t = 0$, so therefore $f(0) = 0$.

In the Kirchner *et al.* (2001) model of transits to a river channel, the above absorbing barrier argument corresponds to saying that the channel water cannot be a source of transits to itself. The only hydrological case might be that of a conceptual raindrop falling onto a recorder within the channel at time $t = 0$. However, this raises the issue again of the recorder raindrop having ability to change catchment mean transit times without ever entering the catchment.

Note on computer simulations

It might be thought that computer simulations could resolve the issue of validity of L-shaped transit time distributions. For example, Fiori and Russo (2008) report a simulated transit time histogram to be well matched by an L-shaped gamma distribution (correlation coefficient R = 0.97). However, computer simulations of probability distributions can only ever be a rediscovery of the assumptions of the simulations. If simulations permit a starting point at the absorbing barrier then a good match to an L-shaped distribution might be achieved, otherwise the simulations will give a poor match to L-shaped distributions when t is near zero.

Alternative 1-parameter distributions with *f(0)* = 0

The exponential distribution is sometimes used in transit time studies as a null hypothesis distribution in the absence of sufficiently detailed data. In fact, any number of alternative 1-parameter distributions with $f(0) = 0$ could be listed for fixed values of a shape parameter. One possibility is the inverse Gaussian distribution with its shape parameter fixed at 1.0:

$$g(t) = \left(\frac{\mu}{2\pi t^3} \right)^{1/2} \exp\left(1 - \frac{1}{2}\left(\frac{t}{\mu} + \frac{\mu}{t} \right) \right) \qquad \textbf{(3)}$$

where μ is the distribution mean. A heavy-tailed option is a particular limit case of the inverse Gaussian distribution:

$$h(t) = \left(\frac{3\xi}{2\pi t^3}\right)^{1/2} \exp\left(-\frac{3\xi}{2t}\right) \qquad (4)$$

with distribution function:

$$H(t) = 2\Phi\left(-\left(\frac{3\xi}{t}\right)^{1/2}\right) \qquad (5)$$

where Φ is the standard normal integral and ξ is the distribution mode.

There is nothing particularly special about the distributions as defined by Equations 3 and 4, but inverse Gaussian distributions do have the property of being derived from transits to an absorbing barrier (Johnson *et al.*, 1994). The distribution given by Equation 3 has some similarities to the exponential distribution (Fig. 1). A simple hydrological conceptual model leading to Equation 4 is described by Bardsley (2017).

Does it matter?

Esoteric considerations like hypothetical raindrops on hypothetical recorders may seem of little relevance to practical hydrological modelling because L-shaped transit time distributions can lead to perfectly reasonable hydrological conclusions and good data fits. Given the choice between a flexible L-shaped transit time distribution and some other flexible distribution with $f(0) = 0$, whichever distribution is selected as the null hypothesis will most likely be the one chosen because there will be insufficient data evidence to reject it. It is entirely possible also that an L-shaped distribution in a given instance might better approximate the true transit time distribution away from $t = 0$.
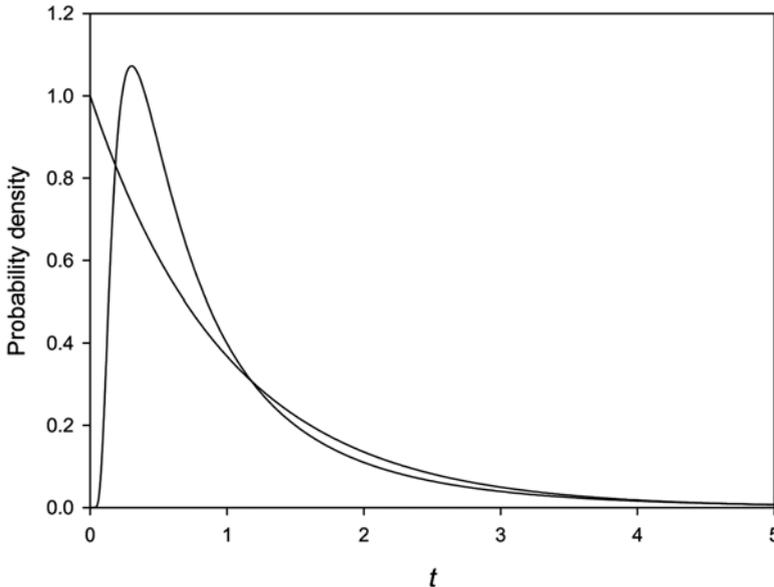

Figure 1 – Comparison between the exponential distribution (mode at t = 0) and the inverse Gaussian distribution as given by Equation 3. Both distributions have mean value = 1.0.

However, little is known of the true forms of transit time distributions, so it may not be good practice to utilise L-shaped distributions knowing they must be incorrect for small values of t, if defining transit time distributions to only include particles that have transited. The philosophy of appropriate distribution selection is well summarised by J.F.C. Kingman in his comment contribution in Folks and Chhikara (1978):

> "Although it is often possible to justify the use of a distribution empirically, simply because it appears to fit the data, it is more satisfactory if the structure of the distribution reflects plausible features of the underlying mechanism."

Discussion and conclusion

There is a large literature on various aspects of transit time distributions in hydrology but the many developing facets of the subject are not relevant to the central theme of suitability of L-shaped probability density functions. In particular, there is no need to take into account whether transit time distributions are transient or stationary.

The L-shaped issue is only relevant to the use of parametric distributions in transit time modelling. For example, there is no concern over the form of nonparametric histograms because the nature of the true distribution near $t = 0$ will be lost in the averaging process over the first bin interval. It is also noted that $f(0) = 0$ implies that all transit time distributions must have at least one mode at $t > 0$. Such modes may or may not be identifiable from field measurements, depending on the local situation and measurement precision.

In the end, the discussion comes down to a question of definitions. From the hydrological point of view, a tracer molecule that is conceptually already 'at' an observation point has no transit history and arguably should not be defined to be part of a distribution of transit times. In that case, transit time distributions cannot be L-shaped.

Finally, all comments here refer to probability density function forms. Practical applications often need only be concerned with cumulative distribution functions (proportion of tracer particles younger than a given age) so the detailed form of the transit time density function near $t = 0$ is not necessarily of consequence for hydrological interpretations.

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