JOURNAL OF HYDROLOGY (N.Z.) Vol. 11, No. 1, 1972

# A MATHEMATICAL MODEL OF THE PHYSICAL PROCESSES OF AN EARTHFLOW

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ABSTRACT

It is postulated that an earthflow behaves as if it were a highly viscous fluid. The consequences of this hypothesis are worked out by means of a mathematical model which predicts velocity profiles through a flow, and rates of erosion. Verification of the hypothesis awaits more experimental data.

## INTRODUCTION

Progress in controlling erosion has been hampered by a fundamental lack of understanding of the processes involved. Consequently, many control measures are not as effective as was hoped, or even fail completely for no obvious reason. The situation is particularly bad in the case of the mudstone earthflows of the East Coast region (Ministry of Works, 1970). These flows can grow rapidly and reach great size. To combat these flows more successfully, the Water and Soil Division, Ministry of Works, has begun a programme of research in an attempt to discover the basic processes governing flows of this nature. Once fundamental processes are understood, more effective control measures aimed at modifying these processes can be devised.

## THE ROLE OF MATHEMATICAL MODELS

A mathematical model is a tool that has a dual role to play:

### 1. *Verification of Process Hypothesis*

The mathematical model does not predict fundamental processes; these processes are hypothesized from experience in the field, and the consequences of this hypothesis are worked out by the model. The model predictions are then compared with reality. If there is poor agreement, the initial hypothesis is proven wrong and must be changed.

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2. *Suggestions for Control Measures*

Once a model is giving correct predictions, then it can be assumed that the fundamental processes have been discovered. The model parameters can then be varied and new predictions made, an attempt being made to find the combination of parameters that gives optimum stability. This will suggest to the soil conservator what he should do, and how effective his measures are likely to be. Further, different parameter combinations could be found that give high although not optimum stability, but which might be easier or more economical to implement.

## THE PROPOSED MODEL

The initial hypothesis that the model will test is that "an earthflow acts as if the mud were a highly viscous fluid". A secondary assumption that is usually made in fluid dynamics and found to hold with a high degree of accuracy, is that the relative velocity of the fluid on any wall is zero (Milne-Thomson, 1960). This paper considers two simple (and simplified) flow geometries to demonstrate the kinds of prediction that these hypotheses give.

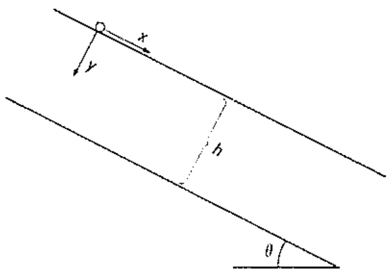


FIG. 1 – Geometry of a two-dimensional flow.

1. *Two-Dimensional Case*

The flow is very wide and is of uniform depth down a slope of angle $\theta$ (Fig. 1). In this case the flow is essentially non-enclosed laterally and side effects can be ignored. Additional assumptions made in this and the next case for the sake of mathematical simplicity are:

(a) The velocity is zero at some depth.
(b) Viscosity and density are constant at all points in the flow.
(c) The mud is a newtonian fluid.
(d) The flow is irrotational, and very slow.

Consider first the case when the mud surface is free, i.e. when it has no covering layer of vegetation. The equation of motion for the viscous fluid in this case is (Milne-Thomson, 1960):

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\eta\left(\frac{\partial^2\mu}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho g\sin\theta$$

where $p$ is the pressure,
$\rho$ is the wet density of the substance,
$\eta$ is the dynamic viscosity of the substance,
$u$ is the $x$ velocity (parallel to the slope).

The boundary conditions are zero velocity at $y=h$ and no shear at $y=0$.

Under steady-state conditions time derivatives vanish and we find

$$u=\frac{\rho g\sin\theta(h^2-y^2)}{2\eta}$$

The velocity profile is parabolic, zero at depth $h$ and maximum at the surface $(=\rho g\sin\theta h^2/2\eta)$ with zero derivative there (Fig. 2).

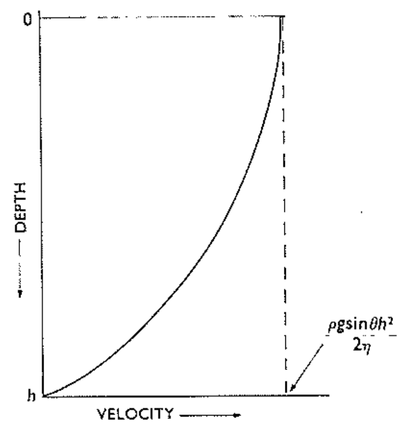


FIG. 2 – Velocity profile for a two-dimensional earthflow with a free surface.

## 2. *Three-Dimensional Case*

The flow is confined to a channel of rectangular cross section, width $a$, depth $b$ (Fig. 3). The relevant equations of motion under steady-state conditions are

$$\frac{\partial p}{\partial y}=\rho g \cos \theta \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-\frac{\rho g \sin \theta}{\eta}$$

where $w$ is the velocity down the slope. The velocity is taken to vanish on the three sides and the normal derivative vanishes on the surface. The second equation above is Poisson's equation, and there exist analytical techniques for solving it given reasonable boundary conditions.

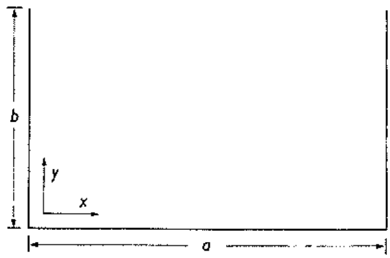


FIG. 3 – Geometry of a three-dimensional flow.

The result in this case is

$$w(x,y)=-\frac{8a^2\rho g \sin \theta}{\pi^3 \eta}$$

$$\sum_{m=1}^{\infty} \frac{\sin\frac{(2m-1)\pi x}{a} \sinh\frac{(2m-1)\pi(2b-y)}{2a} \sinh\frac{(2m-1)\pi y}{2a}}{(2m-1)^3 \cosh\frac{(2m-1)\pi b}{a}}$$

or equivalently

$$w(x,y)=-\frac{32b^2\rho g \sin \theta}{\pi^3 \eta}$$

$$\sum_{m=1}^{\infty} \frac{\sin\frac{(2m-1)\pi y}{2b} \sinh\frac{(2m-1)\pi(a-x)}{4b} \sinh\frac{(2m-1)\pi x}{4b}}{(2m-1)^3 \cosh\frac{(2m-1)\pi a}{4b}}$$

To get something out of this formidable formula, let us consider two special cases.

(a) *Flow much deeper than it is wide.* In this case the solution simplifies to

$$w = -\frac{4a^2\rho g \sin\theta}{\pi^3\eta} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{a}$$

i.e. surface velocity, $w(x,b) = \dfrac{\rho g \sin\theta (x^2 - ax)}{2\eta}$

middle velocity,

$$w(\tfrac{1}{2}a,y) = -\frac{8a^2\rho g \sin\theta}{\eta \cosh(\pi b/a)} \sinh\frac{(2b-y)\pi}{2a} \sinh\frac{\pi y}{2a}$$

Hence the surface velocity is parabolic with its maximum in the middle, while the depth-velocity profile through the middle of the flow is (half) bell-shaped with its maximum at the surface (Fig. 4).

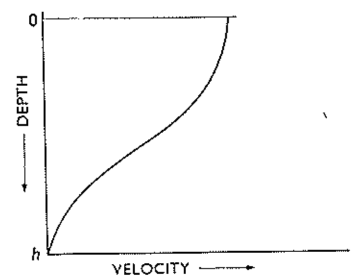


FIG. 4 – Vertical velocity profile for a deep mudflow.

(b) *Flow much wider than it is deep.* Solutions become

surface velocity,

$$w(x,b) = -32b^2\rho g \sin\theta \sinh\frac{\pi(a-x)}{4b} \sinh\frac{\pi x}{4b}$$

middle velocity, $w(\tfrac{1}{2}a,y) = \dfrac{\rho g \sin\theta (y^2 - 2by)}{2\eta}$

In this case the surface velocity profile is bell-shaped with a maximum velocity in the middle, while the depth profile through the middle of the flow is parabolic, with its maximum at the surface (Fig. 5).

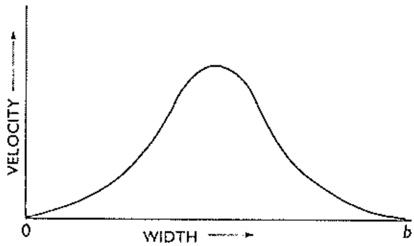


FIG. 5 – Surface velocity profile for a wide flow.

The phenomenon of maximum velocity at the surface may seem a little strange, but this is taken care of if we consider a vegetation layer holding a hardpan of firmer soil perched on the top of the flow. To maintain consistency with the bottom boundary condition (zero relative velocity) we must postulate that the relative velocity between the mud and the top vegetation layer is zero. This, however, still leaves undecided the velocity at which the hardpan will move. The matter can only be resolved by considering a more realistic model in which we consider what is happening at the toe of the flow. Let us assume that the material is not accumulating

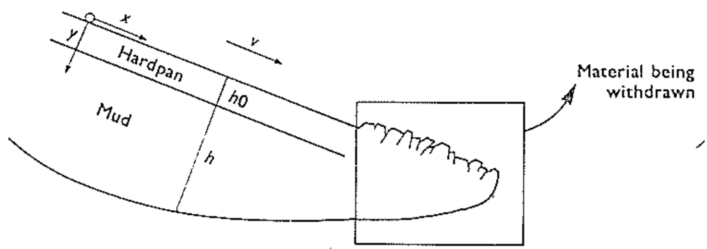


FIG. 6 – The toe of a mudflow with a vegetation cover.

there but is being carried away at the rate of $R$ units of volume per unit width. For the case of a two-dimensional flow (Fig. 6), if $h$ is the depth of the hardpan then

$$v_{h0}+\int_0^h u\,dy=R$$

i.e. the material is arriving at the same rate as it leaves. The new boundary condition that the velocity at $y=0$ is $v$ gives a velocity distribution (Fig. 7) of

$$u=-\frac{\frac{1}{2}\rho g\sin\theta y^2}{\eta}+\left(\frac{\frac{1}{2}\rho g\sin\theta}{\eta}-\frac{v}{h}\right)y+v$$

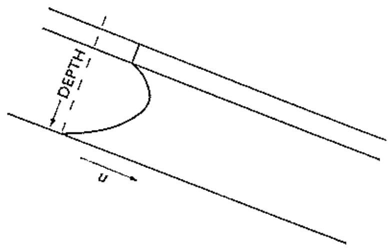


FIG. 7 – Velocity distribution for a mudflow with a vegetation cover.

The velocity profile is now parabolic as for the no-cover case, but with its maximum equal to

$$\frac{\frac{1}{8}\rho g h^2 \sin\theta}{\eta}+\frac{\frac{1}{2}v^2\eta}{\rho g h^2 \sin\theta}+\tfrac{1}{2}v$$

occurring at depth $$\frac{\eta v}{\rho g h \sin\theta}-\tfrac{1}{2}h$$

below the bottom of the pan. It follows that

$$R=\frac{\rho g h^3 \sin\theta}{12\eta}+\tfrac{1}{2}vh$$

This kind of formula connecting rate of erosion to surface velocity, refined for a more realistic model, should give useful information concerning these quantities.

Since surface velocity has an upper bound given by the free surface case considered earliest, the interesting fact that there is a limit to the rate of erosion follows – viz, since

$$v\leqslant\frac{\rho g \sin\theta h^2}{2\eta}$$

then $$R\leqslant\frac{\rho g h^3 \sin\theta}{3\eta}$$

The case in three dimensions poses more problems. However, once either the surface velocity distribution – or better the normal velocity gradient at the surface – is determined, then Poisson's equation above can be solved numerically under these new boundary conditions. The solution will give a velocity distribution in the flow, and the rate of erosion.

## CONCLUSIONS

There is a need for research into the fundamental physical processes of erosion. In this paper it is hypothesized that an earthflow acts as if the flowing material were a highly viscous fluid, and the consequences of this hypothesis have been worked out using a mathematical model. The predictions will be compared with experimental data and the hypothesis modified accordingly. It seems unlikely that the broad hypothesis will be shown to be incorrect, although details will need refining in the light of observations presently being undertaken in the Napier district.

## ACKNOWLEDGMENTS

The author wishes to thank Mr K. Smith for constant feedback on field conditions.

Permission to publish this paper was given by the Commissioner of Works.

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