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UNIT HYDROGRAPH BY DIFFUSION ANALOGY

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ABSTRACT

For a New Zealand stream the hydrograph shape could be modelled according to the diffusion of an exponential decay profile, but this model was ony valid down stream of the point at which the falling limb of the hydrograph was approximately exponential throughout its length. Having established a property of the stream channel analogous to diffusivity, it was possible to synthesize a hydrograph for any point on the stream (subject to the above limitation) given the time of concentration at that point.

Also, from hydrographs at each end of a reach in a stream, the Muskingham method could be used to synthesize a corresponding hydrograph at any other point within or down stream of the original reach, given the time of concentration at that point.

Hydrographs synthesized according to the diffusion-analogy method and the Muskingham method were in good agreement.

INTRODUCTION

The Puketurua experimental basin, which has been described by the N.Z. Ministry of Works (1968), has three gauging stations, all on the main channel, namely Puketurua with a basin area of 2.48 km^2, Pukewaenga with a basin area of 0.441 km^2 and Pukeiti with a basin area of 0.0144 km^2.

Unit hydrographs were calculated for each gauging point from storms on 30 August 1967 and 5, 7, 13 and 25 December 1968. Each hydrograph was immediately preceded by a period of intense rainfall, with little or no rainfall during the period of surface runoff. Characteristics of the storms and hydrograph rises are given in Table 1.

Mean unit hydrographs are plotted semilogarithmically in Fig. 1 and linearly in Fig. 2; the extremities of the I-bars in Fig. 1 represent two-thirds confidence limits for the mean ordinate based on a sample of five. In each figure the curves designated I, II and III represent unit hydrographs for Puketurua, Pukewaenga and

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Pukeiti respectively, each unit hydrograph having an area of 1 unit of runoff by 1 hour.

TABLE 1 — Characteristics of storms and hydrograph rises analysed for unit hydrographs. (Values shown are mean, with range in brackets.)

	Puketurua	*Pukewaenga*	*Pukeiti*
Rainfall intensity (mm/h)	21.7 (11.5–28.3)	19.8 (11.5–23.8)	23.2 (17.7–37.9)
Peak surface runoff (mm/h)	3.9 (2.3–5.3)	12.7 (8.9–18.9)	33.8 (17.9–50.9)
Surface runoff volume (mm)	9.0 (3.7–11.8)	11.3 (8.7–14.0)	21.4 (13.1–39.7)
Runoff coeffiicent	0.40 (0.17–0.52)	0.52 (0.50–0.54)	0.88 (0.72–1.05)
Ground water flow at hydrograph peak (mm/h)	0.7 (0.5–1.3)	0.4 (0.3–0.6)	0.3 (0.2–0.4)

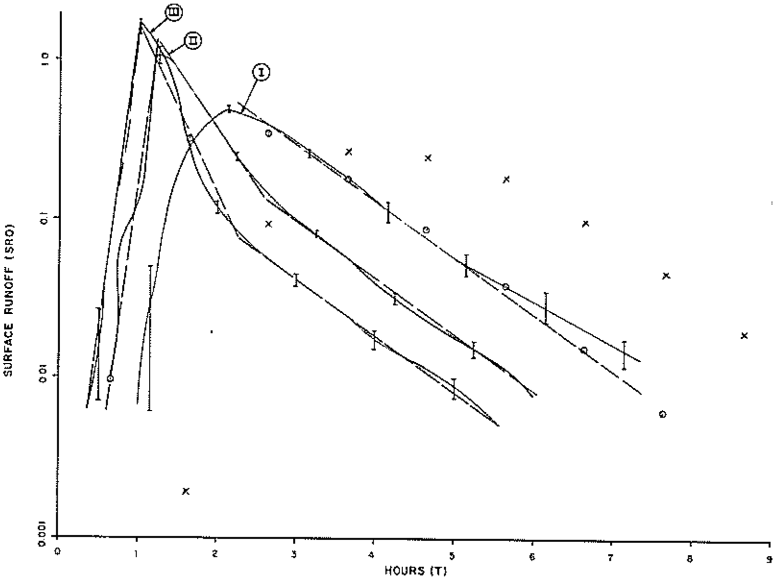

FIG. 1 — Unit hydrographs at Pukeiti (III), Pukewaenga (II) and Puketurua (I); semilogarithmic plot.

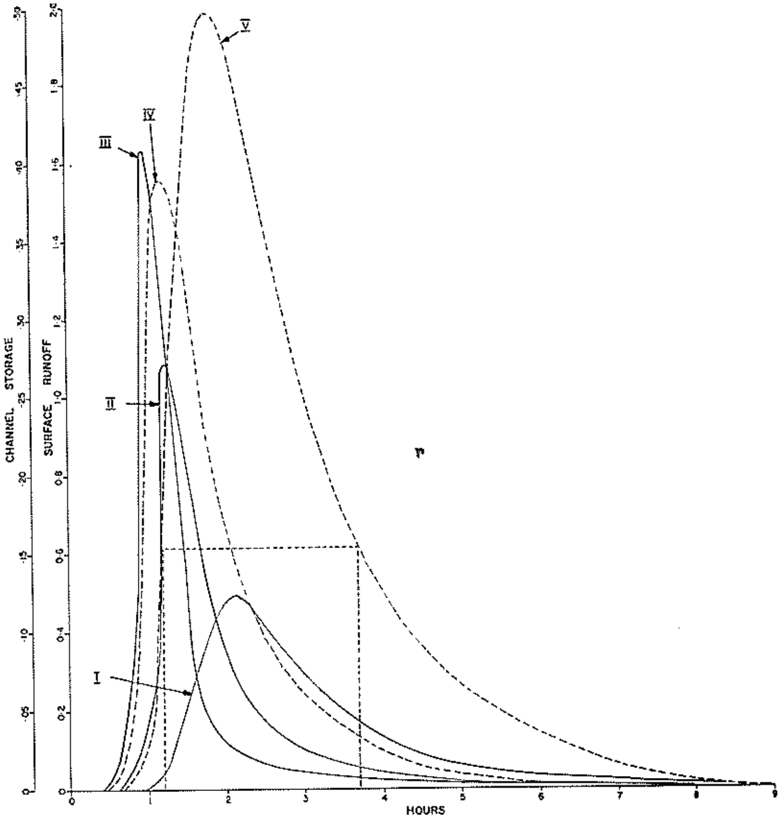

FIG. 2 — Unit hydrographs at Pukeiti (III), Pukewaenga (II) and Puketurua (I), and corresponding channel storages for Pukeiti-Pukewaenga reach (IV) and Pukewaenga-Puketurua reach (V); linear plot.

DIFFUSION-ANALOGY MODEL

In Fig. 1 the equations of the straight broken lines which approximate the parts of the hydrograph which are close to linear are:

Puketurua	SRO$=\exp(1.20-0.81T)$	$T>2.26$
Pukewaenga	SRO$=\exp(-10.72+8.78T)$	$T<1.25$
	$=\exp(2.40-1.69T)$	$1.25<T<2.60$
	$=\exp(-0.10-0.81T)$	$T>2.60$
Pukeiti	SRO$=\exp(-8.28+8.78T)$	$T<1.00$
	$=\exp(2.94-2.45T)$	$1.00<T<2.26$
	$=\exp(-0.76-0.81T)$	$T>2.26$

For all hydrographs the slope of the lower parts of the falling limbs were the same, but for hydrographs for gauging points progressively up stream the slopes of the higher parts of the falling limb became progressively steeper.

The steeper parts of the falling limbs of the Pukeiti and Pukewaenga unit hydrographs are attributed to overland flow, but this component is not detectable in the Puketurua unit hydrograph.

For the purpose of constructing a model it was assumed that at some section in the stream between Pukewaenga and Puketurua the falling limb of the hydrograph would be approximately linear throughout its length, with neither the overland flow steepening as exhibited at Pukewaenga, nor the flattening of the peak as at Puketurua. From this hypothetical section down stream, the hydrograph shape will be modified in a manner typified by the shape at Puketurua.

El-Baroudi (1969) provides a model for the dispersion of an exponential decay wave according to Fick's second law of diffusion. The equation of the model unit hydrograph is similar to El-Baroudi's equation for the dispersion of a soluble tracer in its passage through a continuous-flow two-cell tank, and is:

$$Q_t = \tfrac{1}{2} C e^{M+N} \operatorname{erfc}[M/(2\sqrt{N}) + \sqrt{N}]$$

where erfc is the complementary error function, given by

$$\operatorname{erfc}(p/\sqrt{2}) = 2 - \sqrt{(2/\pi)} \int_{-\infty}^{p} \exp(-\tfrac{1}{2}t^2)\,dt$$

and $M = K^2Dt$,
$N = K(T-t)$,
Q_t = surface runoff at time t (mm/h),
i = rainfall intensity (mm/h),
C = runoff coefficient,
K = inverse duration of rainfall (h^{-1}),
T = time of concentration (h),
t = time (h),
D is analogous to diffusivity (h).

For the Puketurua 1-hour unit hydrograph the variables take the following values:

$K = 1.0$ for 1-hour rainfall period,
$iC = 1.0$, since the product of iC and $1/K$ is unity for a unit hydrograph,
T = time to peak minus half rainfall duration = 1.65.
D was fitted by trial to 0.13.

Ordinates of the model unit hydrograph fitted to the measured Puketurua unit hydrograph are shown as small circles in Fig. 1.

Using a value of D of 0.13 a unit hydrograph was synthesized for $T = 3.65$, that is 2 hours greater than for Puketurua. This is tabulated in Table 3 and shown by small crosses in Fig. 1.

MUSKINGHAM METHOD

In the absence of gauging stations down stream of Puketurua, the only method of checking the synthetic unit hydrograph for $T = 3.65$ is to compare it with a synthetic unit hydrograph generated by some other procedure. The Muskingham method, which is discussed by Langbein (1942), can provide a check hydrograph.

Given hydrographs at two sections on a stream, the Muskingham method first analyses the hydrographs to yield two parameters of the reach between the two sections; these parameters may be described as the attenuation factor and the lag factor, which roughly represent the change in shape in the hydrograph in the reach, and the time for the hydrograph to pass through the reach. Subsequently, the Muskingham method provides for the modification of any other up-stream hydrograph to synthesize a corresponding hydrograph at the down-stream section.

Further, given hydrographs at two sections on a stream, it is possible to synthesize a hydrograph for any other section between or down stream of these two sections. For this it is necessary to assume that the attenuation factor remains constant throughout the reach in which the original sections and the new section are located. Also, the time of concentration at the new section must be known, from which the lag factor can be calculated as the difference between the times of concentration at the new section and at the closer up stream of the two original sections. Having obtained the attenuation and lag factors appropriate to the reach between the new section and the closer up stream of the two original sections, a hydrograph may be calculated for the new section by the normal routine.

Normally the attenuation factor is determined by a trial-and-error semigraphical procedure, but an alternative calculation procedure is quicker and at least as accurate. First, the storage within the reach is calculated as accumulated inflow at the upper end of the reach minus outflow at the lower end. Storage rises from zero to a maximum (when inflow equals outflow) then falls to zero. Any given storage S_{12} (less than the maximum) is associated with two sets of inflow and outflow, one set on the rising limb of the storage curve (I_1 and O_1) and one set on the falling limb (I_2 and O_2). The attenuation factor X may be calculated as:

$$X = \frac{O_2 - Q_1}{O_2 - O_1 + I_1 - I_2} \qquad (1)$$

since

$$S_{12}/Y = XI_1 + (1-X)O_1 = XI_2 + (1-X)O_2 \qquad (2)$$

In Fig. 2, curve IV represents storage within the Pukeiti-

Pukewaenga reach and curve V represents storage within the Pukewaenga-Puketurua reach, both as a proportion of total unit surface runoff.

The rectangle inscribed within curve V illustrates the method of finding X and Y, in this case for the Pukewaenga-Puketurua reach and for $S=0.15$.

A horizontal line at $S=0.15$ is drawn to intersect curve V twice and verticals are drawn from the two intersections of curve V and $S=0.15$, each vertical to intersect the Pukewaenga hydrograph (II) and the Puketurua hydrograph (I). The vertical from the rising limb of S intersects the Pukewaenga (inflow) hydrograph at $I_1=0.60$, and the Puketurua (outflow) hydrograph at $O_1=0.04$. The vertical from the falling limb of S intersects the Pukewaenga hydrograph at $I_2=0.05$ and the Puketurua hydrograph at $O_2=0.16$.

The value of X is then calculated from equation (1):

$$X=\frac{O_2-O_1}{O_2-O_1+I_1-I_2}=\frac{0.16-0.04}{0.16-0.04+0.60-0.05}=0.18$$

and the value of Y from equation (2):

$$\begin{aligned} Y &= S_{12}/[XI_1+(1-X)O_1] \\ &= 0.15/(0.18\times 0.60+0.82\times 0.04)=1.07 \end{aligned}$$

Values for X and Y were calculated by the above procedure at S intervals of 0.05 (see Table 2).

TABLE 2 — Muskingham attenuation factor (X) and lag factor (Y) at varying channel storage.

	Pukeiti-Pukewaenga		*Pukewaenga-Puketurua*	
S	*X*	*Y*	*X*	*Y*
0.05	0.06	0.64	0.08	1.32
0.10	0.17	0.62	0.19	1.16
0.15	0.13	0.53	0.18	1.07
0.20	0.18	0.51	0.14	0.97
0.25	0.23	0.49	0.18	0.98
0.30	0.29	0.50	0.20	1.01
0.35	—	—	0.24	1.02
0.40	—	—	0.28	1.05
Mean:	0.18	0.55	0.19	1.07

For both reaches X increased significantly with S, showing that the conventional semigraphical method for finding X, which requires X to be constant with S, was not applicable to this stream.

On the other hand, *Y* was steadier with varying *S*, and mean values of *Y* (0.55 and 1.07) were roughly comparable to the lag between peaks in the reaches (0.25 and 0.9 hours respectively).

In order to synthesize a unit hydrograph at a point down stream of Puketurua, *X* was set to 0.20 by extrapolation of *X* for the upper reaches and *Y* was set to 2.0, the required delay between peaks.

Table 3 shows unit hydrographs synthesized by the diffusion-analogy method and the Muskingham method for a time of concentration of 3.65 hours, i.e. Puketurua plus 2 hours.

TABE 3 — Two synthetic unit hydrographs on Puketura Stream for 3.65 hours time of concentration.

Time (hours)	*Diffusion analogy*	*Muskingham method*
0.65	0.00	0.00
1.65	0.00	0.02
2.65	0.10	0.17
3.65	0.27	0.27
4.65	0.25	0.23
5.65	0.19	0.15
6.65	0.10	0.10
7.65	0.05	0.06
8.65	0.02	0.02
9.65	0.00	0.01

CONCLUSIONS

For the Puketurua Stream, unit hydrographs were synthesized for a certain time of concentration by both the diffusion-analogy method and the Muskingham method. The two synthetic unit hydrographs were very similar.

The diffusion-analogy method requires only one unit hydrograph to establish a characteristic of the stream analogous to diffusivity, from which a unit hydrograph for any given time of concentration can be synthesized for the stream. A limitation is that the original unit hydrograph must be — and all synthetic unit hydrographs are — convex upwards when plotted on log-normal paper.

The Muskingham method requires hydrographs at two sections on the stream to establish a factor, known as the attenuation factor, from which a unit hydrograph for any given time of concentration may be synthesized. The synthetic hydrograph should be for a point between or down stream of the two sections.

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