NON-UNIFORM FLOW IN OPEN CHANNELS A GRAPHICAL SOLUTION

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SUMMARY

A graphical method is given for construction of the surface profile for flow at constant discharge and varying depth in a channel of any cross section. The construction is used to determine the relationship between the depths at the ends of the channel.

The slope of the tangent to the profile at any place along the channel is shown to be a readily determined function of depth alone. A small increment of depth is considered and it is assumed that the profile through this depth change can be replaced by the tangent. Starting from a known water surface elevation at one end of the channel and using the function which defines the slope of the tangent, a sequence of tangents is constructed and the surface elevation at the other end of the channel is found.

The construction is repeated for several initial depths, so that a curve may be plotted showing the depth at one end of the channel as a function of the depth at the other end.

INTRODUCTION

The ubiquity of steady non-uniform flow in open channels establishes the need for convenient and accurate means of computation. Problems which arise frequently are determination of surface profiles and of depth-discharge relationships. The first of these needs no comment. The second may be discussed in terms of the expression $Q = f(y_1, y_2)$ in which Q is the discharge, y_1 is the depth at the upstream end and y_2 the depth at the downstream end of a reach with more or less constant cross section and bed slope (Fig. 1).

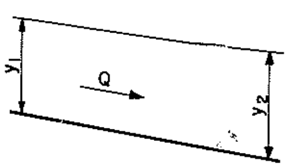

FIG. 1

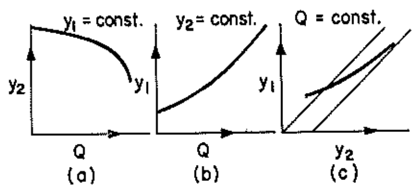

FIG. 2

With y_1 held constant Q becomes a function of y_2 and with y_2 held constant Q is a function of y_1. With Q held constant y_1 can be displayed as a function of y_2 and a family of y_1 versus y_2

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curves for a set of values of Q displays the whole relationship (Figs 2a, b, c). Detailed descriptions of the curves are given by Bakhmeteff (1932).

To find any of these relationships it is necessary to integrate the varied flow equation. This may be done by using a step by step arithmetical process, or by using tables of the varied flow function in the methods of Bakhmeteff (1932), von Seggern (1949), and Chow (1959). All of these require extensive arithmetical labour. The work is tedious and tiring. One's attention is apt to wander and mistakes are frequently made. Moreover, in those methods one loses touch with the engineering problem while the computation is in progress.

In this paper, a graphical method is described in which arithmetical labour is reduced to a minimum. The integration is achieved by construction of a surface profile — a matter of direct interest to the engineer.

CONVEYANCE AND SECTION FACTOR

Use is made of two concepts introduced by Bakhmeteff (1932), the conveyance and the section factor. It is assumed that the reach under consideration is prismatic — e.g., a man-made canal — or is sufficiently constant in shape for a mean cross section and bed slope to be determined. The conveyance is seen to be a measure of the discharge capacity of the cross section and the section factor to be related to flow at critical depth as follows:

Conveyance

The Chezy formula gives $V = C\sqrt{mi}$ in which V is mean velocity; C is Chezy's coefficient; m is hydraulic mean radius; i is slope of total energy line.

∴ The discharge is given by $Q = CA\sqrt{mi}$ where A is area of cross section.

Rearranging gives $\frac{Q}{\sqrt{i}} = CA\sqrt{m}$ and the conveyance K is defined by $K = CA\sqrt{m}$. Using the Manning formula $C = \frac{1.49}{n} m^{1/6}$ in which n is a channel roughness coefficient, $K = \frac{1.49}{n} Am^{2/3}$ and $\frac{Q}{\sqrt{i}} = K$. In particular, $\frac{Q}{\sqrt{S_0}} = K_0$ in which S_0 is the bottom slope and K_0 is the conveyance for the discharge Q to flow at constant depth — i.e., with $i = S_0$.

The significance of this equation is the separation of the variables which depend on the depth of flow and the cross section (K) from the discharge and slope. The particular value $K_0 = \frac{Q}{\sqrt{S_0}}$ leads to y_0 the depth for uniform flow.

Section Factor

The section factor is defined by $M = A\sqrt{\frac{A}{b}}$ where b = surface width.

The relationship of the section factor to critical depth can be seen from the specific energy:

Specific energy $E_s = y + \frac{V^2}{2g} = y + \frac{Q^2}{2gA^2}$ where y is depth of flow.

For critical depth $\frac{dE_s}{dy} = O$ i.e. $1 - \frac{Q^2}{gA^3}\frac{dA}{dy} = O$ and

since $\frac{dA}{dy} = b$ then $\frac{A^3}{b} = \frac{Q^2}{g}$ or $A\sqrt{\frac{A}{b}} = \frac{Q}{\sqrt{g}}$

For any value of Q, then, there is a particular value of $M(= A\sqrt{\frac{A}{b}})$ corresponding to critical flow. This value of M is M_c and, since M is a function of y alone, M_c gives y_c, the critical depth.

It is an essential preliminary to this method that K and M be plotted as functions of y. In columns 1-10 of Table 1 and in Fig. 3 this has been done for the canal used as an example.

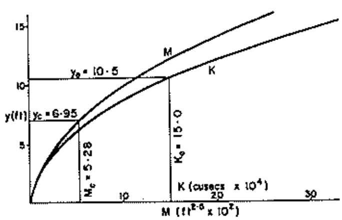


FIG. 3

TABLE 1

$$K_o = \frac{Q}{\sqrt{S_o}} = 15.0 \times 10^4 \quad \therefore y_o = 10.5 \text{ ft} \quad M_c = \frac{Q}{\sqrt{g}} = 528$$

$$\therefore y_c = 6.95 \text{ ft}$$

	y	20 + 2y	A	P	m	$m^{2/3}$	K x 10^{-4}	b	$\sqrt{\frac{A}{b}}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(1)	2	24	48	28 9	1.66	1.40	0.672	28	1.31
(2)	4	28	112	37.9	2.96	2.06	2.31	36	1.77
(3)	6	32	192	46.8	4.11	2.57	4.93	44	2.09
(4)	8	36	288	55.8	5.16	2.99	8.62	52	2.36
(5)	10	40	400	64.7	6.18	3.37	13.50	60	2.58
(6)	12	44	528	73.6	7.17	3.72	19.65	68	2.79
(7)	14	48	672	82.5	8.14	4.05	27.2	76	2 98
(8)	16	52	832	91.5	9.09	4.36	36.3	84	3.15
(9)	7.00						6.6		
(10)	7.50						7.55		
(11)	9.00						10.9		
(12)	11.00						16.4		
(13)	13.00						23.3		
(14)	15.00						31.45		
(15)	7.10						6.8		
(16)	7.375						7.3		

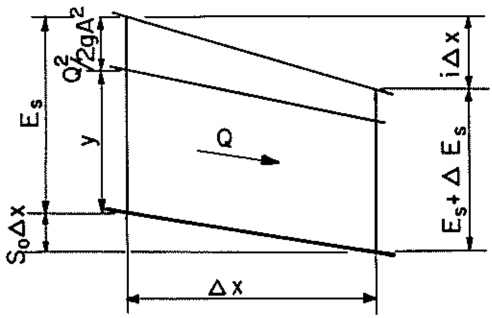
$K = 100\ Am^{2/3}$ $\qquad A = y(20 + 2y)$ $\qquad P = 20 + 4.47y$

$m = \frac{A}{P}$ $\qquad M = A\sqrt{\frac{A}{b}}$ $\qquad b = 20 + 4y$

M	$(\frac{K_0}{K})^2$	$(\frac{M_c}{M})^2$	$1-(\frac{K_0}{K})^2$	$1-(\frac{M_c}{M})^2$	$\frac{dy}{dx} \times 10^4$	$L\frac{dy}{dx}$
(10)	(11)	(12)	(13)	(14)	(15)	(16)
62.9						
198						
401						
678	3.02	0.607	—2.02	0.393	—20.6	—5.15
1031	1.24	0.261	—0.240	0.739	—1.30	—0.325
1470	0.580	0.128	+0.420	0.872	+1.93	+0.482
2000	0.303	0.0697	+0.697	0.9303	+3.00	+0.750
2620	0.170	0.0450	+0.830	0.955	+3.48	+0.870
530	5.17	1.00	—4.17	0	—∞	—∞
600	3.95	0.773	—2.95	0.227	—52.0	—13.0
842	1.89	0.392	—0.89	0.608	—5.85	—1.46
1237	0.835	0.182	+0.165	0.818	+0.806	+0.202
1777	0.414	0.0882	+0.586	0.912	+2.57	+0.641
2293	0.226	0.0530	+0.774	0.947	+3.24	+0.810
550	4.86	0.921	—3.86	0.079	—195	
585	4.22	0.813	—3.22	0.187	—83.9	

VARIED FLOW EQUATION

The varied flow equation is a non-linear first order differential equation for the depth of flow in terms of distance. It may be derived as follows (Fig. 4):



Total energy line slope i (+ve)
Invert slope S_o (+ve when invert falls in direction of flow.)

FIG. 4

$$(E_s + \Delta E_s) + i\Delta x = S_o\Delta x + E_s$$

$$\therefore \Delta E_s = (S_o - i)\ \Delta x$$

$$\text{But } \Delta E_s = \frac{dE_s}{dy} x\Delta y = (1 - \frac{Q^2 b}{gA^3})\ \Delta y$$

$$\therefore (1 - \frac{Q^2}{g}\frac{b}{A^3})\ \Delta y = (S_o - i)\ \Delta x$$

$$\text{and } \frac{dy}{dx} = S_o \frac{1 - \frac{i}{S_o}}{1 - \frac{Q^2}{g}\frac{b}{A^3}} = S_o \frac{1 - (\frac{K_o}{K})^2}{1 - (\frac{M_c}{M})^2}$$

With Q fixed, K_o and M_c are both determined and $\frac{dy}{dx}$ is a function of y alone.

GRAPHICAL SOLUTION

In essence, the graphical solution is simply a matter of defining the slopes of tangents to the water surface profile in a convenient way and drawing short lengths of these tangents through successive small increments of depth to construct the surface profile.

Let FA (Fig. 5) be the invert of a reach of length L and LG the water surface profile. Consider a short length of the surface profile in the vicinity of the point P. Through this short length the depth changes from $y - \frac{\Delta y}{2}$ to $y + \frac{\Delta y}{2}$ and it is assumed that this element of the profile may be replaced by the tangent at P.

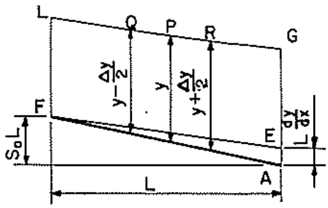


FIG. 5

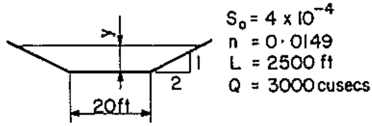


FIG. 6

The depth at P fixes $\frac{dy}{dx}$ so that $L\frac{dy}{dx}$ can be determined. This is the height that the tangent would rise relative to the invert in the length of the reach. If AE is set out at the downstream end of the reach equal to $L\frac{dy}{dx}$ FE defineds the slope of the tangent. A line QR through P and parallel to FE is, therefore, an element of the profile through P.

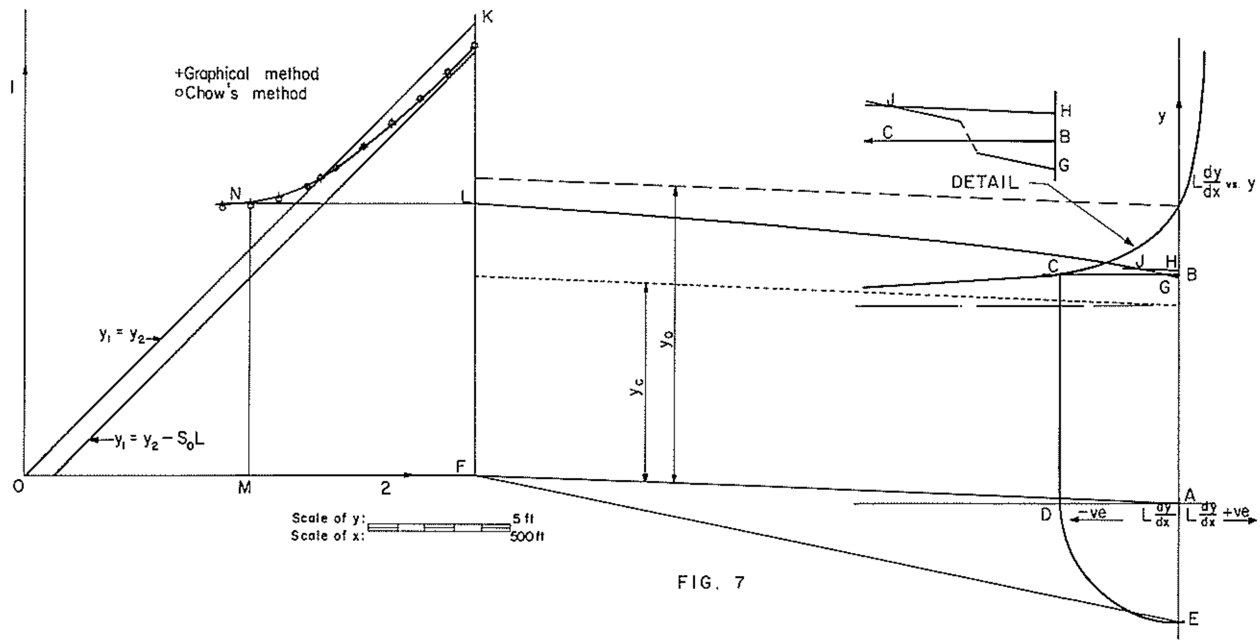
This analysis is the justification for the graphical construction. The objective of the method is to construct a curve of y_1 vs y_2 for a chosen value of Q. The following notes demonstrate the method of describing its application to a particular problem (see Fig. 6).

(i) Calculate columns 1-10 for rows 1-8 in Table 1 leading to values of K and M which are plotted on Fig. 3.

(ii) Find K_0 $(= \frac{Q}{\sqrt{S_0}})$ and M_c $(= \frac{Q}{\sqrt{g}})$ and hence, from Fig. 3 obtain y_0 and y_c.

(iii) Complete rows 4-8 of Table 1 by calculating columns 11-16 leading to $L\frac{dy}{dx}$. If necessary, additional values of $L\frac{dy}{dx}$ can be determined as in rows 9-14 of Table 1 using values of K and M from Fig. 3.

(iv) Draw an elevation of the reach using a suitable scale distortion, say 100:1 (see Fig. 7). Draw lines parallel to the invert FA at uniform depth y_0 and critical lepth y_c. At the upstream end, set out the axes 01, 02 on which y_1, y_2 will be plotted and draw lines $y_1 = y_2$ (representing uniform



+Graphical method
○Chow's method
I
K
N
L
$y_1 = y_2$
$y_1 = y_2 - S_oL$
O
M
2
F
Scale of y: 5 ft
Scale of x: 500 ft
y_c
y_o
DETAIL
J
H
C
B
G
y
$L\frac{dy}{dx}$ vs. y
A
D
−ve
$L\frac{dy}{dx}$
+ve
E

FIG. 7

flow) and $y_1 = y_2 - S_0L$ (representing a horizontal water surface).

At the downstream end, plot $L\frac{dy}{dx}$ horizontally (negative values to the left and positive values to the right and using the same scale as for y) against y. Note that $L\frac{dy}{dx} = O$ when $y = y_0$ and $L\frac{dy}{dx} = -\infty$ when $y = y_c$. There is another branch of the curve between $y = O$ and $y = y_c$, but it is of no interest in this example. This branch of the curve would be required if super-critical flows were to be investigated.

(v) Choose a value of y_2, say 8.00 ft. The procedure described now gives an element of the profile for the first depth increment to y = 8.25 ft. Set out AB = 8.125 ft (the mean of 8.00 and 8.25) at the downstream end.) Project B horizontally to C on the curve of $L\frac{dy}{dx}$, project C vertically to D on the axis y = O and swing an arc with centre A and radius AD to E vertically below A. (E is below A because $L\frac{dy}{dx}$ is negative).

Set out AG = 8.00 ft and AH = 8.25 ft. Draw GJ parallel to FE (F is at the upstream end of the invert) and HJ parallel to FA, the invert, to intersect GJ in J. Then GJ is the first element of the profile.

One is free to choose any size of increment, and 0.25 ft has been chosen here as an example. Clearly, small increments will be necessary where the slope is changing rapidly, i.e. near critical depth, and larger increments will be acceptable elsewhere. Always, however, the slope of the tangent at mid-height of the increment is used to define the element of the profile.

(vi) Repeat the procedure for successive increments of depth until the profile intersects FK at the upstream end of the reach in L. The depth FL is the value of y_1 corresponding to $y_2 = 8.00$ ft.

(vii) Erect an ordinate MN at $y_2 = 8.00$ ft (axes 01,02) and project L horizontally to N on this ordinate.

(viii) Repeat the whole procedure for different initial values of y_2 and draw a fair curve through the points N.

COMMENT

(i) It will be desirable to draw a profile for $y_2 = y_c$ — this is the lowest profile for sub-critical flow. The graphical procedure cannot be used for the first one or two increments because of the very large values of $L \frac{dy}{dx}$. For these intervals it is necessary to calculate $\Delta x = \Delta y \ / \ \frac{dy}{dx}$ and to set out the horizontal distances.

In the problem discussed above, $y_c = 6.95$ ft and the first two increments for the profile with $y_2 = y_c$ are $y = 6.95 — 7.25$ ft and $y = 7.25 — 7.50$ ft. The mean depths for these increments are 7.10 ft and 7.375 ft respectively and calculation of the appropriate values of $\frac{dy}{dx}$ is shown in lines 15 and 16 of Table 1. Then, for the first increment

$$\Delta x = \Delta y \ / \ \frac{dy}{dx} = - \frac{0.30}{195 \times 10^{-4}} = 15 \text{ ft}$$

and for the second increment

$$\Delta x = - \frac{0.25}{83.9 \times 10^{-4}} = - 30 \text{ ft}$$

By the third increment ($y = 7.50 — 7.75$ ft) the tangents are flat enough to be handled conveniently by the graphical procedure.

(ii) Obviously, the uniform depth profile will be used to give a point on the y_1 versus y_2 curve.

(iii) When the depth is near uniform depth, the intersections which define successive points on the profile are very acute This difficulty is overcome by careful work and scale distortion.

(iv) When constructing the profile it must be remembered that y is measured vertically from the invert and x is measured horizontally.

CONCLUSIONS

The graphical method is justified by: (a) its accuracy and versatility, (b) the reduction of arithmetical labour, and (c) the fact that the engineering problem is continuously in view.

A check on the accuracy of the method has been made by using Chow's method to solve the same problem. The computed paints are shown on Fig. 7 and the agreement is very satisfactory.

The graphical method is versatile because the curves of K and M versus y can be determined for any shape of cross section.

Provided the cross section can be drawn to scale, the area for any chosen depth can be measured with a planimeter and the perimeter and breadth with a scale. K and M can be calculated using these measured characteristics. The methods using the varied flow function tables also have this versatility, but they require use of approximations for K and M — viz

$$\left(\frac{K}{K_0}\right)^2 = \left(\frac{y}{y_0}\right)^N \text{ and } \left(\frac{M}{M_c}\right)^2 = \left(\frac{y}{y_c}\right)^M$$

These approximations are not necessary when the graphical construction is used.

The reduced arithmetical labour is obvious. The number of computed values of the various quantities is small and arithmetical interpolation between tabulated values is not required. Interpolation is one of the baneful aspects of using the varied flow function.

Continual emphasis of the engineering outlook is provided by constructing profiles. It also makes solution of the varied flow equation very straightforward. These features make the method attractive to the author who hopes that others will also find it attractive.

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LIST OF SYMBOLS

y — depth of flow
y_1 — depth at upstream end
y_2 — depth at downstream end
y_0 — depth for uniform flow
y_c — critical depth
Δy — increment of depth
x — horizontal distance
Δx — increment of distance
L — length of reach
A — area of cross section
m — hydraulic mean radius
P — wetted perimeter
b — surface width
i — slope of total energy line
S_0 — slope of channel bed
C — Chezy's coefficient
n — channel roughness coefficient
K — conveyance
K_0 — conveyance for depth y_0
M — section factor or an exponent
M_c — section factor for depth y_c
N — an exponent
Q — discharge
g — gravitational acceleration
V — mean velocity
E_s — specific energy